Tokenomics: Dynamic Adoption and Valuation*

Lin William Cong[†] Ye Li[§] Neng Wang[‡]

First Draft: February 4, 2018
This Draft: September 10, 2018

Abstract

We provide a dynamic asset-pricing model of (crypto-)tokens on (blockchain-based) platforms, and highlight their roles on endogenous user adoption. Tokens intermediate transactions on decentralized networks, and their trading creates an inter-temporal complementarity among users, generating a feedback loop between token valuation and platform adoption. Consequently, tokens capitalize future platform growth, accelerate adoption, reduce user-base volatility, and can improve welfare. Equilibrium token price increases non-linearly in platform productivity, user heterogeneity, and endogenous network size. The model also produces explosive growth of user base after an initial period of dormant adoption, accompanied by a run-up of token price volatility. We further discuss how our framework can be used to discuss cryptocurrency supply, token competition, and pricing assets under network externality.

JEL Classification: C73, F43, E42, L86

Keywords: Bitcoin, Blockchain, Cryptocurrency, Digital Currency, ICOs, FinTech, Network Effect, Platforms, Tokens.

---

*The authors thank Philip Bond, Jaime Casassus, Tom Ding, Alex Frankel, Zhiguo He, Dirk Jenter, Andrew Karolyi, Yongjin Kim, Pietro Veronesi, Johan Walden, Randall Wright, Yizhou Xiao, and seminar and conference participants at Chicago Booth, London Finance Theory Group Summer Conference, FinanceUC 14th International Conference, City University of Hong Kong International Finance Conference, ESSFM Gerzensee Asset Pricing Workshop, Emerging Trends in Entrepreneurial Finance Conference, Norwegian School of Economics, Rome Junior Finance Conference, Stanford SITE, and the Shanghai Forum for helpful comments. Cong gratefully acknowledges financial support from the Center for Research in Security Prices.

[†]University of Chicago Booth School of Business. E-mail: will.cong@chicagobooth.edu

[§]Ohio State University. E-mail: li.8935@osu.edu

[‡]Columbia Business School and NBER. E-mail: neng.wang@columbia.edu

1 Introduction

Blockchain-based cryptocurrencies and tokens have taken the world by storm. According to CoinMarketCap.com, the entire cryptocurrency market capitalization has also grown from around US$20 billion to around US$600 billion over last year, with active trading and uses; virtually unknown a year ago, initial coin offerings (ICOs) have also attracted more attention than the conventional IPOs, raising 3.5 billion in more than 200 deals in 2017 alone, according to CoinSchedule. In order to draw a line between reckless speculation and financial innovation, and understand how tokens should be regulated, it is important to first understand how cryptocurrencies or tokens (henceforth generically referred to as "token") derive value and the roles they play in the development and adoption of the virtual economy.

To this end, we develop the first dynamic model of a virtual economy with endogenous user adoption and native tokens that facilitate transactions and business operations. We anchor token valuation on the fundamental productivity of the (blockchain-based) platform, and demonstrate how tokens derive value as an exchangeable asset with limited supply that users hold to derive utility available solely on the platform. We then pin down the dynamics of token price as the solution to a second-order ODE. We are also the first to highlight two important roles of tokens in business development (fundraising included). First, the expected price appreciation makes tokens attractive to early users, allowing them capitalize future growth of the platform and accelerating adoption. Second, because the expected price appreciation diminishes as the platform technology matures and more users adopt, the endogenous token price change moderates the volatility of user base caused by to productivity shocks.

Specifically, we consider a continuous-time economy with a continuum of agents who differ in their needs to conduct transactions on the blockchain. We broadly interpret transaction as including not only typical money transfer (e.g., on the Bitcoin blockchain) but also signing smart contracts (e.g., on the Ethereum blockchain). Accordingly, we model agents' gain from blockchain transaction as a flow utility that depends on agent-specific transaction needs, the size of blockchain community, and the current productivity of blockchain platform ("productivity" broadly interpreted) that loads on exogenous shocks. Very importantly, the larger the community is, the more surplus can be realized through trades among agents on the blockchain (i.e., higher flow utility of tokens). Exogenous shocks to productivity can be broadly interpreted as shocks to the general usefulness of the platform, technological changes, or regulatory shocks such as the ban on cryptocurrency trading by several governments.

In our model, agents make a two-step decision: (1) whether to incur a participation cost to meet potential trade counterparties (i.e., to join the community); (2) how many tokens to hold, which depends on both blockchain trade surplus ("transaction motive") and the expected future token price ("investment motive"). A key innovation of our model is that not only does one user's adoption exhibit externality on others, but the investment motive also introduces an inter-temporal complementarity in user adoption. We model productivity as a geometric Brownian motion, thus exogenous shocks have permanent impact on the level of productivity. A positive shock directly increases the user base today by increasing the flow utility of holding tokens. At the same time, agents expect more users to join the community in future, which leads to a stronger future demand for tokens and thus token price appreciation. The investment motive then creates a stronger demand for tokens today and greater adoption. This dynamic feedback mechanism leads to a token pricing formula that reflects agents' expectation of future popularity of the platform, in addition to productivity, and user heterogeneity.

We characterize the non-degenerate Markov equilibrium with platform productivity as the state variable, and derives the token valuation as the solution to a second-order ordinary differential equation with boundary conditions that reflect platform fundamentals instead of bubbles. Akin to many equilibrium models that feature interaction between financial markets and the real economy, the financial side of our model is the endogenous price of tokens, whereas the real side is the size of user base that determines the benefits (the utility flow) of agents who trade on the blockchain. Token price affects user adoption through the expected price appreciation, while user base affects token price through its impact on flow utility and agents' token demand. This mutual feedback naturally triggers a question: how a platform with embedded tokens differs from one without?

We therefore compare the endogenous S-curve of adoption of a platform in three economies: the first-best, tokenless economy (where agents use dollars or other media of exchange), and tokenized economy, all having exactly the same process of productivity dynamics. We find that without tokens, user adoption is below the first-best level which entails full adoption as long as the platform productivity is above a threshold. Tokens can improve welfare because the user base grows faster and reaches full adoption faster. That said, a caveat is that the investment motive can lead to over-adoption in the very early stage of the platform, not to mention that tokens can accelerate the demise of a bad platform whose productivity drifts downward—as agents forecast a smaller user base in the future, they shun away from holding tokens with expected price depreciation. In sum, embedding tokens on a platform front-loads the prospect of platform, and can either accelerate adoption or precipitate abandonment depending on the expected evolution of the fundamentals (i.e., productivity in our model).

Furthermore, we show that introducing tokens can reduce user base volatility, making it less sensitive to productivity shocks. The key driver is again the agents' investment motive—their decision to participate depends on their expectation of future token price appreciation. Consider a negative shock that reduces the flow utility, and thus user adoption. This direct negative effect is mitigated by an indirect effect through token price: A lower adoption now means more agents can be brought onto the platform in future. Agents' expected stronger token price appreciation therefore induces them to adopt and hold tokens. Similarly, a positive productivity shock increases adoption by increasing the flow utility. However, as the pool of potential newcomers shrinks, the expected token price appreciation declines, discouraging agents from joining the platform and holding tokens.

Having clarified the roles of tokens analytically, we calibrate our model to existing data. In addition to illustrating our key mechanism, the quantitative exercise also helps understand several empirical patterns in token price: the endogenous user adoption can generate run-ups in token valuation and volatility.

Finally, we extend our discussion along several dimensions to demonstrate the model's flexibility in accommodating richer features. Specifically, we illustrate in the appendix how endogenizing productivity growth can further strengthen the economic channels we highlight, how time-varying systematic risk of tokens can produce a sharp rise and fall of token price under rational expectations, and how our model can be used to analyze cryptocurrency competition and the design of state-contingent token supply.

Overall, our model sheds light on the pricing of cryptocurrencies and tokens in peer-to-peer networks that include but is not restricted to permissioned and permissionless blockchains.[1]

[1]To be precise, the networks in question should be viewed as complete networks with economy of scale, which is different from incomplete networks that many recent studies focus on (e.g., Acemoglu, Carvalho,

Many digital currencies and tokens, payment-focused or not, have been introduced and associated with platforms and virtual economies: Linden dollar for the game Second Life, WoW Gold for the game World of Warcraft, Facebook Credits, Q-coins for Tencent QQ, Amazon coins, to name a few.[2] Our model offers a pricing formula and reveals how introducing native tokens benefits users and accelerates adoption. While the Blockchain technology certainly gives platforms unprecedented flexibility and commitment power in introducing native tokens and designing their attributes, our model potentially applies to other trusted platforms or virtual systems such as email protocols and online social network, and adds new insights to asset pricing and macro models with network externality.

Literature Review. Our paper contributes to the emerging literature on blockchains and cryptocurrencies. Among early studies, Cong and He (2018) examine informational issues in generating decentralized consensus with implications on industrial organization; Biais, Bisiere, Bouvard, and Casamatta (2017) and Saleh (2017) analyze the mining or minting games through Proof-of-Work and Proof-of-Stake; Easley, O'Hara, and Basu (2017), Huberman, Leshno, and Moallemi (2017), and Cong, He, and Li (2018) study miners' compensation, organization, and microstructure; Harvey (2016) briefly surveys the mechanics and applications of crypto-finance; Yermack (2017) and Cao, Cong, and Yang (2018) evaluates the impact of the technology on corporate governance and financial reporting.

To be clear, the concepts of digital currency and distributed ledger have been separately developed previous and Nakamoto's innovation lies in combining them to enable large-scale application (Narayanan and Clark (2017)): embedding a native currency into a blockchain system helps incentivize record-keepers (e.g., miners in protocols using proof-of-work) to form decentralized consensus, which in turn prevents double-spending (Nakamoto (2008)). Our paper analyzes the impact of native tokens on user adoption, highlighting the demand side of blockchain-generated consensus.

Specifically, we focus on the valuation of cryptocurrencies (tokens) under endogenous user adoption in a *dynamic* framework that highlights inter-temporal feedback effects. In con-

Ozdaglar, and Tahbaz-Salehi (2012)).

[2]Even before the heated debate on cryptocurrencies, economists, and commentators were already raising questions such as "Could a gigantic nonsovereign like Facebook someday launch a real currency to compete with the dollar, euro, yen and the like?" (Yglesias (2012)). Gans and Halaburda (2015) provides an insightful introduction on how payment systems and platforms are related.

trast, other models in the literature are static. For example, Gans and Halaburda (2015) is among the earliest studies on platform-specific virtual currencies and users' network effects. Ciaian, Rajcaniova, and Kancs (2016) test quantity theory of money using Bitcoin data assuming exogenous user demand for Bitcoins. Fernández-Villaverde and Sanches (2016) and Gandal and Halaburda (2014) consider the competition among alternative cryptocurrencies. Pagnotta and Buraschi (2018) studies the pricing of Bitcoins under exogenous network and adoption. Closer to our paper is Athey, Parashkevov, Sarukkai, and Xia (2016) that emphasizes agents' learning of a binary technology quality and decision to use bitcoins for money transfer, but does not model contemporaneous and intertemporal user-base externality.

Several contemporaneous models analyze cryptocurrencies in the context of initial coin offerings (ICOs). Li and Mann (2018) demonstrate that staged coin offerings mitigate coordination issues; Sockin and Xiong (2018) study households' purchase of an indivisible cryptocurrency as membership certificate that enables them to match and trade with others; Catalini and Gans (2018) study entrepreneurs' discretionary pricing of tokens to fund start-ups; Chod and Lyandres (2018) discuss the risk diversification benefit of ICOs; most recently, Bakos and Halaburda (2018) compare the adoption acceleration benefit of tokens we highlight with traditional user subsidy through VC capital.

We differ in our emphasis of the role of crypto-tokens as media of exchange in decentralized virtual economy, and their dynamic effects on endogenous user adoption.[3] We differ also in allowing agent heterogeneity and divisible holdings of tokens, and study the joint dynamics of token valuation and user adoption.[4] Unlike many studies focusing on permissionless blockchains maintained by decentralized miners, our framework does not rely on the specific mechanism for consensus generation, and consequently applies equally to permissioned blockchains or platforms owned by trusted third parties with network effect of user adoption.

We organize the remainder of the article as follows. Section 2 sets up the model. Section 3 solves the dynamic equilibrium, and derives the token valuation formula. We choose

[3]Also related are discussions on the design of cryptocurrencies/tokens and platforms, such as Gans and Halaburda (2015), Halaburda and Sarvary (2016), Chiu and Wong (2015), and Chiu and Koeppl (2017). Our framework can directly tie the protocol-based design on token supplies to token valuation and adoption, enabling us to evaluate various design objectives.

[4]Bakos and Halaburda (2018) also discuss user adoption, but in a setting without technological uncertainty or user heterogeneity. Like Sockin and Xiong (2018), the model features a two-period setup in which tokens serve as platform membership.

parameter values in Section 4 and discuss the implications on the model's quantitative performances. Section 5 highlights the roles of tokens in user adoption and compares tokenized, tokenless, and the first-best economies. We analyze how the token price evolves with the user base, endogenous volatility, and token risk premium in Section 6. Section 7 provides additional institutional background on crypto-currency and crypto-tokens. Section 8 concludes. The appendix contains all the proofs, additional analysis, and model extensions.

2 A Model of Tokenized Economy

Consider a continuous-time economy where a unit measure of agents conduct peer-to-peer transactions and realize trade surpluses on a platform, e.g., a blockchain. A generic good serves as the numeraire ("dollar"). We first set up and solve the model under the risk-neutral measure. In Section 4, we calibrate the model under the physical measure.[5]

2.1 Blockchain Technology and Agent Heterogeneity

Platform transaction surplus. The blockchain platform allows agents to conduct peer-to-peer transactions. These transactions are settled via a medium of exchange, which can either be the numeraire (e.g., dollar) or the native token for this blockchain. We use $x_{i,t}$ to denote the value of agent i's holdings in the unit of the numeraire. These holdings facilitate transactions on the platform and generate a flow of utility over dt given by

$$x_{i,t}^{1-\alpha} \left(N_t A_t e^{u_i}\right)^{\alpha} dt, \tag{1}$$

where N_t is the platform user base, A_t measures blockchain productivity, u_i captures agent i's specific needs for blockchain transactions, and $\alpha \in (0,1)$ is a constant.[6]

[5] No arbitrage and complete market imply a unique probability measure—the risk-neutral measure—under which agents discount future cash flow using the risk-free rate.

[6] Our results are qualitatively robust to alternative specifications that feature decreasing total return i.e., $(x_{i,t})^{1-\alpha-\gamma} \left(N_t A_t e^{u_i}\right)^{\alpha}$ with $\gamma > 0$. Results are available upon request.

The platform productivity, A_t, evolves according to a geometric Brownian motion:

$$\frac{dA_t}{A_t} = \mu^A dt + \sigma^A dZ_t^A. \tag{2}$$

We focus on the case of a promising yet risky platform, i.e., $\mu^A > 0$ and $\sigma^A > 0$. We interpret A_t broadly. A positive shock to A_t reflects technological advances in cryptography and computation, favorable regulatory or policy changes, growing users' interests, and increasing variety of activities feasible on the platform.

The transaction surplus depends on N_t, the total measure of agents that join the platform (i.e., $x_{i,t} > 0$). Introducing it captures the network externality among users. For example, it reflects the ease to find trading or contracting counterparties in a large community.

User heterogeneity and adoption. We assume that agents' transaction needs, u_i, are heterogeneous. Let $G(u)$ denote the cross-sectional cumulative distribution function of u_i and $g(u)$ be its density function, which is assumed to be continuously differentiable over a finite support $[\underline{U}, \overline{U}]$. Next, we provide a few blockchain examples to interpret u_i.

For payment blockchains (e.g., Ripple and Bitcoin), a high value of u_i reflects agent i's urge to conduct a transaction (e.g., an international remittance and a purchase of drugs). For smart-contracting blockchains (e.g., Ethereum), u_i captures agent i's project productivity. For decentralized computation (e.g., Dfinity) and data storage (e.g., Filecoin) applications, u_i corresponds to the need for secure and fast access to computing power and data.

To join the platform and realize the transaction surplus, an agent has to incur a flow cost ϕdt. For example, transacting on the platform takes effort and attention. At any time t, agents may choose not to participate and then collect no transaction surplus. Therefore, agents with sufficient high u_i choose to join the platform, while agents with sufficiently low u_i do not participate.

2.2 Tokens, Agents' Problem, and Equilibrium

Tokens. In what follows, we focus on the joint dynamics of token valuation and user adoption on platforms requiring native tokens as the medium of exchange, i.e.,

$$x_{i,t} = P_t k_{i,t}, \tag{3}$$

where P_t is the unit price of token in terms of the numeraire and $k_{i,t}$ is the units of token.[7]

We conjecture and verify that the equilibrium price dynamics is a diffusion process,

$$dP_t = P_t \mu_t^P dt + P_t \sigma_t^P dZ_t^A \,, \tag{4}$$

where μ_t^P and σ_t^P are endogenously determined.

Throughout the paper, we use capital letters for aggregate and price variables that individuals take as given, and lower-case letters for individual-level variables.

Agent's problem. Let $y_{i,t}$ denote agent i's *cumulative* profits from blockchain activities. Agent i then maximizes life-time utility under the risk-neutral measure,

$$\mathbb{E}\left(\int_{t=0}^{\infty} e^{-rt} dy_{i,t}\right) \,, \tag{5}$$

where we can write the incremental profit $dy_{i,t}$ as follows:

$$dy_{i,t} = \max\left\{0, \max_{k_{i,t}>0}\left[\left(P_t k_{i,t}\right)^{1-\alpha}\left(N_t A_t e^{u_i}\right)^{\alpha} dt + k_{i,t}\mathbb{E}_t\left[dP_t\right] - \phi dt - P_t k_{i,t} r dt\right]\right\}. \tag{6}$$

Here, the outer "max" operator reflects agent i's option to leave the platform and obtain zero profit, and the inner "max" operator reflects agent i's optimal choice of $k_{i,t}$.

Inside the inner max operator are four terms that add up to give the incremental profits from participating in the blockchain transaction. The first term corresponds to the blockchain trade surplus given in (1). The second term is the expected capital gains from

[7] The dollar value of inputs ($P_t k_{i,t}$), instead of $k_{i,t}$ alone, shows up in the surplus flow to facilitate the comparison between platforms with and without tokens. It is also motivated by that fact that the economic value of blockchain trades depends on the numeraise value of real goods and services that are transacted in tokens. Our results are qualitatively similar in the alternative specification with only $k_{i,t}$ (instead of $P_t k_{i,t}$) in the trade surplus,

holding $k_{i,t}$ units of tokens, where $\mathbb{E}_t[dP_t] = P_t \mu_t^P dt$. Users care about the sum of the on-chain transaction surplus and the expected token appreciation given by the first two terms in (6). The third term is the participation cost and the last term is the financing cost of holding $k_{i,t}$ units of tokens.

It is worth emphasizing that in our tokenized economy, agents must hold tokens for at least the instant to complete transactions and derive utility flows. This holding period exposes users to token price change over dt, but is important for transactions on blockchains (e.g., by addressing the double-spending problem). We provide institutional details and examples in Section 7.

The Markov equilibrium. We study a Markov equilibrium with A_t as the only state variable. Token supply is fixed at a constant M.[8] The market clearing condition is

$$M = \int_{i \in [0,1]} k_{i,t} di, \tag{7}$$

where for those who do not participate, $k_{i,t} = 0$.

Definition 1. *A Markov equilibrium with state variable A_t is described by agents' decisions and equilibrium token price such that the token market clearing condition given by Equation (7) holds and agents optimally decide to participate (or not) and choose token holdings.*

3 Dynamic Equilibrium of Adoption and Valuation

In this section, we solve the Markov equilibrium. The aim is to solve endogenous variables, such as the user base, N_t, users' token holdings, $k_{i,t}$, and token price P_t, as functions of the state variable, A_t. First, we study agents' decision to participate and hold tokens, given A_t and agents' expectation of token price change μ_t^P. Then we complete the solution by solving the token price dynamics (and in particular, μ_t^P as a function of A_t). Each step ends with a summarizing proposition.

[8] This is consistent with many ICOs that fix the supply of tokens. Because resources for business operations on-chain are all discussed in real terms, we can simply normalize M to one due to money neutrality.

Token demand and user base. Conditional on joining the platform, agent i chooses the optimal token holdings, $k_{i,t}^{*}$, by the first order condition,

$$(1-\alpha)\left(\frac{N_t A_t e^{u_i}}{P_t k_{i,t}^{*}}\right)^{\alpha}+\mu_t^P=r, \tag{8}$$

that is the sum of *marginal* transaction surplus on the platform and expected token price change equals r. Rearranging the equation, we have the optimal token holdings, $k_{i,t}^{*}$,

$$k_{i,t}^{*}=\frac{N_t A_t e^{u_i}}{P_t}\left(\frac{1-\alpha}{r-\mu_t^P}\right)^{\frac{1}{\alpha}}. \tag{9}$$

$k_{i,t}^{*}$ has several properties. First, agents hold more token when the common productivity, A_t, or agent-specific transaction need, u_i, is high, and also when the user base, N_t, is larger because it is easier to conduct trades on the platform. Equation (9) reflects an investment motive to hold tokens, that is $k_{i,t}^{*}$ increases in the expected token appreciation, μ_t^P.

Using $k_{i,t}^{*}$, we solve the maximized profits conditional on participation,

$$N_t A_t e^{u_i}\alpha\left(\frac{1-\alpha}{r-\mu_t^P}\right)^{\frac{1-\alpha}{\alpha}}-\phi. \tag{10}$$

Apparently, agent i chooses to hold tokens (i.e., join the platform) if the expression is non-negative. Taking logarithm of it, we solve $\underline{u}_t$, the user participation threshold,

$$\underline{u}_t=\underline{u}\left(N_t;A_t,\mu_t^P\right)=-\ln\left(N_t\right)+\ln\left(\frac{\phi}{A_t\alpha}\right)-\left(\frac{1-\alpha}{\alpha}\right)\ln\left(\frac{1-\alpha}{r-\mu_t^P}\right), \tag{11}$$

so we know that agents with $u_i \geq \underline{u}_t$ participate, that is,

$$N_t=1-G\left(\underline{u}_t\right). \tag{12}$$

$\underline{u}_t$ is decreasing in A_t because a more productive platform attracts more users. The adoption threshold also decreases if agents expect token price to increase more (i.e., higher μ_t^P).

Together, Equations (11) and (12) solve N_t. Given A_t and μ_t^P, we note that zero adoption is always a solution. We focus on the non-degenerate case, i.e., $N_t > 0$, and consider the properties of a *response* function $R\left(n;A_t,\mu_t^P\right)$ that maps a hypothetical value of N_t, say n, to

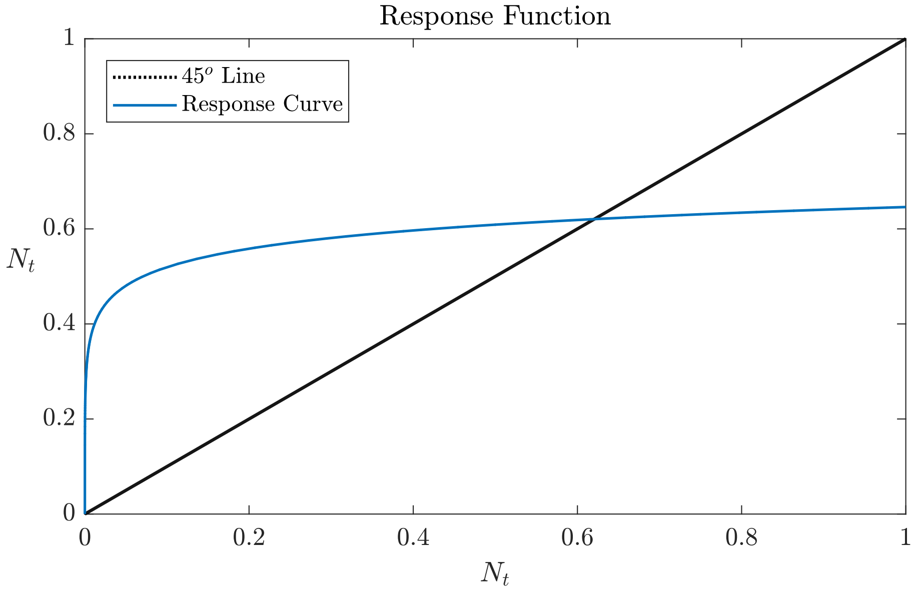

Figure 1: Determining User Base.

the measure of agents who choose to join the community after knowing $N_t = n$. As depicted in Figure 1, the response curve originates from zero (the degenerate case). In the Appendix, first, we show that given μ_t^P, there exists a threshold $\underline{A}\left(\mu_t^P\right)$ such that for $A_t < \underline{A}\left(\mu_t^P\right)$, a non-degenerate solution does not exist, because the response curve never crosses the 45^o line. Then we prove that when $A_t \geq \underline{A}\left(\mu_t^P\right)$, the response curve crosses the 45^o line exactly once (and from above) under the assumption of increasing hazard rate of $g(u)$, .

Proposition 1 (**Token Demand and User Base**). *Given μ_t^P and A_t ($> \underline{A}\left(\mu_t^P\right)$, defined in the Appendix), there exists a $N_t > 0$ that is a unique non-degenerate solution of Equations (11) and (12) if $g(u)$ has an increasing hazard rate.*[9] *N_t is increasing in μ_t^P and A_t.*

Agent i's optimal token holdings conditional on participation, $k_{i,t}^*$, is given by Equation (9). $k_{i,t}^*$ increases in A_t, μ_t^P, u_i, and N_t, and decreases in P_t.

[9]Increasing hazard rate means $\frac{g(u)}{1-G(u)}$ is increasing in u, which is equivalent to $1 - G(u)$ being log-concave. This is a common assumption, for example, in the mechanism design literature to avoid the technically complicated "ironing" of virtual values. We may rule out the degenerate equilibrium using a different functional form of trade surplus: $(P_t k_{i,t})^{1-\alpha} (A_t e^{u_i})^{\alpha} dt + (P_t k_{i,t})^{1-\alpha} N_t^{\alpha} dt$, i.e., with N_t entering the surplus in an additive form. Under this specification, there are always participating agents whose u_i is high enough. The multiplicative specification (Cobb-Douglas) of surplus is for analytical convenience.

Token Pricing and Complete Solution. Next, we solve token price. First, define the *participants' aggregate transaction need* as

$$S_t := \int_{\underline{u}_t}^{\overline{U}} e^u g(u)\, du. \tag{13}$$

S_t is the integral of e^{u_i} of participating agents. Substituting in agents' optimal token holdings into the token market clearing condition (Equation (7), we have

$$P_t = \frac{N_t S_t A_t}{M} \left(\frac{1-\alpha}{r - \mu_t^P} \right)^{\frac{1}{\alpha}}. \tag{14}$$

The token price increases in N_t – the larger the user base is, the higher trade surplus individual participants can realize by holding tokens, and stronger the token demand. The price-to-user base ratio increases in the blockchain productivity, the expected price appreciation, and network participants' aggregate transaction need, while decreases in token supply M.[10] The formula reflects certain observations by practitioners, such as incorporating DAA (daily active addresses) and NVT Ratio (market cap to daily transaction volume) in token valuation framework, but instead of heuristically aggregating such inputs into a pricing formula, we solve *both* token pricing and user adoption as an equilibrium outcome.[11]

The token pricing formula also implies a differential equation that solves $P(A_t)$, i.e., the token price as a function of state variable A_t. By Itô's lemma, μ_t^P is equal to $\left(\frac{dP_t/P_t}{dA_t/A_t}\right)\mu^A + \frac{1}{2}\frac{d^2P_t/P_t}{dA_t^2/A_t^2}\left(\sigma^A\right)^2$. From Proposition 1, we can solve $\underline{u}_t$ and N_t, which only depends on A_t and μ_t^P (i.e., the first and second derivatives of $P(A_t)$). Substituting these variables into the token pricing formula (Equation (14)), we have the right-hand side depends only on A_t, $P(A_t)$, and the first and second derivatives of $P(A_t)$ through μ_t^P.[12] Rearranging the token

[10]Our model contributes to the asset pricing literature by rationalizing the commonly used valuation-to-user base ratio in the technology industry, especially popular for valuing internet companies with user network effects. But note that the asset we price is a blockchain token, not equity stakes of firms.

[11]See, for example, *Today's Crypto Asset Valuation Frameworks* by Ashley Lannquist at Blockchain at Berkeley and Haas FinTech.

[12]Since $\underline{u}_t$ decreases in μ_t^P, the right-hand side of the token market clearing condition increases in μ_t^P, so we can uniquely pin down μ_t^P given A_t and P_t. Therefore, there exists a unique mapping from A_t, $P(A_t)$, and $P'(A_t)$ to $P''(A_t)$.

pricing formula, we have a second-order ordinary differential equation (ODE) for $P(A_t)$:

$$\frac{\left(\sigma^A\right)^2}{2}\frac{d^2P_t/P_t}{dA_t^2/A_t^2}+\mu^A\left(\frac{dP_t/P_t}{dA_t/A_t}\right)+(1-\alpha)\left(\frac{N_tS_tA_t}{MP_t}\right)^\alpha-r=0. \tag{15}$$

By imposing proper boundary conditions, we can solve for $P(A_t)$. The first is

$$\lim_{A_t\to 0}P(A_t)=0, \tag{16}$$

so that when the platform is not productive any more, token price collapses to zero. The second involves that as A_t increases, token price after full adoption ($N_t=1$) behaves according to

$$\overline{P}(A_t)=\frac{\overline{S}A_t}{M}\left(\frac{1-\alpha}{r-\mu^A}\right)^{\frac{1}{\alpha}}, \tag{17}$$

where $\overline{S}\equiv\int_{\underline{U}}^{\overline{U}}e^u g(u)\,du$ is the aggregate of all agents' e^{u_i}. In analogy with Equation (14), Equation (17) is essentially the "Gordon Growth Formula" in our setting with N_t, S_t, and μ_t^P replaced by 1, $\overline{S}$, and μ^A respectively. At $\underline{u}_t=\underline{U}$, we require value matching and smooth pasting conditions:

$$P(A^*)=\overline{P}(A^*)\quad and\quad P'(A^*)=\overline{P}'(A^*). \tag{18}$$

Together, these conditions rule out the degenerate equilibrium of $P(A_t)=0$ for any $A_t>0$.[13]

Proposition 2 (Token Pricing Formula and the Markov Equilibrium Solution). *In a Markov equilibrium with A_t as the state variable, the token price, $P(A_t)$, is solved by the ODE from the pricing formula (Equation (14)) and boundary conditions given by Equation (16) and (18).*[14] *Given the token price dynamics, agents' optimal token holdings and participation decision (and the user base) are solved in Proposition 1.*

Figure 2 summarizes the key economic mechanism that follows from all the results thus

[13]Ruling out such a degenerate equilibrium is non-trivial because the dollar value of tokens, i.e., $P_tk_{i,t}$, enters into the trade surplus, so when $P_t=0$, the flow utility of token is zero. However, given Equation (18), agents expect token price to be positive in the far future where A_t is sufficiently large. Hence reasoning backward instant by instant, token price stays positive so that arbitrage opportunities (i.e., a jump of token price from zero to positive within an instant of dt) do not exist.

[14]The ODE given by Equation (14) satisfies all the conditions in Theorems 4.17 and 4.18 in Jackson (1968) that guarantee the existence and uniqueness of solution. The existence of a unique equilibrium distinguishes our paper from studies such as Sockin and Xiong (2018) that focus on equilibrium multiplicity, and allows us to highlight dynamics of adoption and valuation.

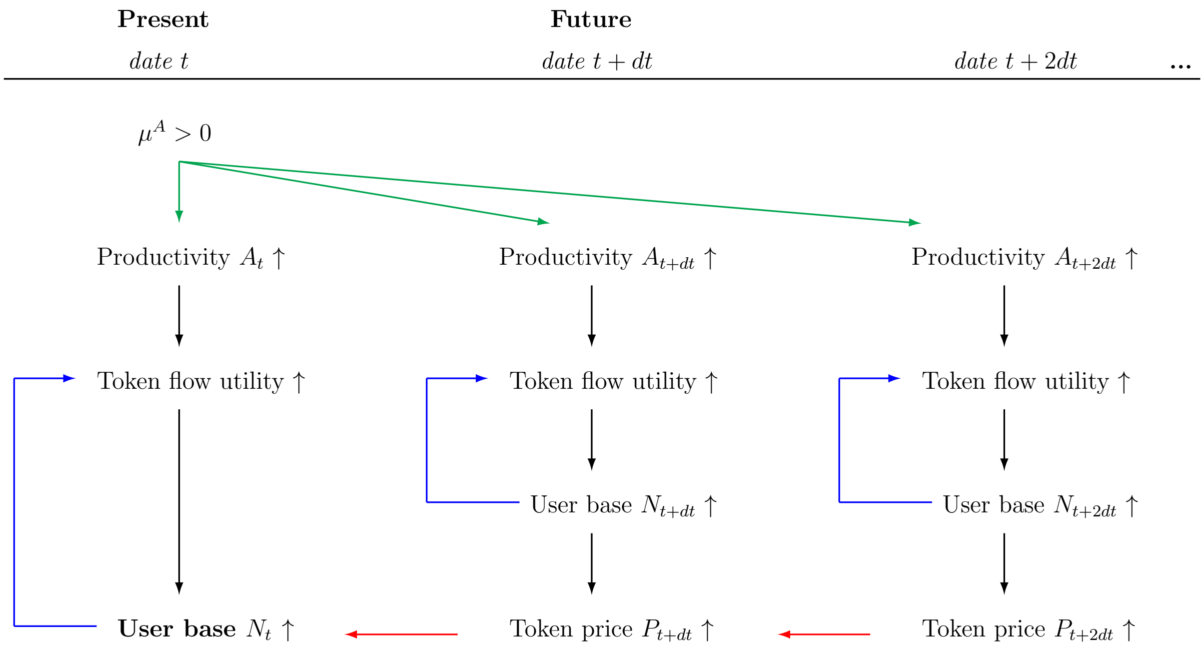

Figure 2: **The Economic Mechanism.** The green arrows point to the increases of the current and future (expected) levels of productivity A, which lead to higher flow utilities of tokens, and in turn, larger user bases N (black arrows). The blue arrows show that increases in user base result in even higher flow utility due to the contemporaneous user-base externality. Finally, more users push up the token prices P in future dates, which feed into a current expectation of token price appreciation and greater adoption (red arrows).

far, where the blue, black, and red arrows show respectively the user-base externality, the transaction motive of token holdings, and the investment motive of token holdings.

4 Parameter Values

We examine the model implications with numerical solutions. Here we describe how we set parameter values under the physical measure so that the model generates key patterns in the data of token price and user base.

SDF and the physical measure. So far we have worked under the risk-neutral measure. To relate our model to data, we discuss price and adoption dynamics under the physical measure. To this end, we assume that agents' risk preference is given by a stochastic discount factor ("SDF") Λ satisfying

$$\frac{d\Lambda_t}{\Lambda_t} = -rdt - \eta d\hat{Z}_t^{\Lambda}, \tag{19}$$

where r is the risk-free rate and η is the price of risk for systematic shock $\hat{Z}_t^\Lambda$ under the physical measure.[15] Let dZ_t^A denote the SDF shock under the risk-neutral measure, so

$$dZ_t^\Lambda = d\hat{Z}^\Lambda + \eta dt, \tag{20}$$

by the Girsanov Theorem. Throughout this paper, we use "ˆ" indicate the physical measure. A unique SDF gives the unique Arrow-Debreu security prices, implying a complete market.

Let ρ denote the instantaneous correlation between the SDF shock and the blockchain productivity shock. The usefulness of a particular platform evolves with the economy, as agents discover new ways to utilize the technology, which in turn depends on the progress of complementary technologies. As aforementioned, macro and regulatory events affect the usage of a blockchain platform. The crypto beta from ρ is priced under the physical measure, generating a link between token price fluctuation and expected return.

Under the physical measure, A_t has the following law of motion, $dA_t = \hat{\mu}^A A_t dt + \sigma^A A_t d\hat{Z}_t^A$, where using Girsanov Theorem, we know that $\hat{\mu}^A$ is equal to $\mu^A + \eta\rho\sigma^A$, and $d\hat{Z}_t^A$ is the Brownian productivity shock under the physical measure, given by $d\hat{Z}_t^A = dZ_t^A - \eta\rho dt$.

Parameter values. Our choice of parameter values is guided by token price and blockchain user-base dynamics from July 2010 and April 2018. We set one unit of time in the model to be one year4 . Since we fix the token supply at M, the token price P_t completely drives the market capitalization ($P_t M$). We map P_t to the *aggregate* market capitalization of major cryptocurrencies.[16] Since we study a representative token economy, focusing on the aggregate market averages out idiosyncratic movements due to specificalities of token protocols.

We collect the number of active user addresses for these cryptocurrencies, and map the aggregate number to N_t. Since the beginning month of our sample is unlikely to be the initial date of blockchain application, we choose to map the maximum number of active addresses (in December 2017) to $N_t = 0.5$, and scale the number in other months by dividing it by

[15]The SDF in the form of Equation (19) can be generated from a consumption-based asset pricing model (see, for example, Chen (2010)).

[16]We include all (sixteen) cryptocurrencies with complete market cap and active address information on bitinfocharts.com, namely, AUR (Auroracoin), BCH (Bitcoin Cash), BLK (BlackCoin), BTC (Bitcoin), BTG (Bitcoin Gold), DASH (Dashcoin), DOGE (DOGEcoin), ETC (Ethereum Classic), ETH (Ethereum), FTC (Feathercoin), LTC (Litecoin), NMC (Namecoin), NVC (Novacoin), PPC (Peercoin), RDD (Reddcoin), VTC (Vertcoin). They represent more than 2/3 of the entire crypto market.

this maximum number. We focus on the model performances in the early stage of adoption, i.e., $N_t \in [\underline{N}, 0.5]$. $\underline{N}$ will be explained later together with Figure 3. We take a truncated Normal distribution for u_i[17]. Parameter values are chosen such that the model generates the following data patterns: (1) the growth of N_t over time; (2) the evolution of the growth *rate* of N_t; (3) the co-movement between P_t and N_t; (4) the dynamics of user base volatility. We later juxtapose our model-generated results along these dimensions together with the data.

The key parameters for the equilibrium dynamics of N_t and P_t are μ^A, σ^A, α, and θ. First, μ^A and σ^A determine the time scale, i.e. how fast A_t, N_t and P_t grow over time, because the instantaneous expected growth rate of A_t is $\hat{\mu}^A = \mu^A + \eta\rho\sigma^A$ under the physical measure. η is set to 1, in line with the maximum Sharpe ratio of the U.S. stock market. ρ is set to 1, a conservative value relative to the beta of technology sector (Pástor and Veronesi (2009)). We set $\mu^A = 2\%$ and $\sigma^A = 200\%$, so that $\hat{\mu}^A = 202\%$ and N_t grows from 0.0001 ($\underline{N}$) to 0.5 during the eight-year period of our data sample. More intuition is explained below.

σ^A contributes most of A_t's growth under the physical measure. As previously discussed, we interpret A_t broadly, including technological advances, regulatory changes, and the variety of activities feasible on the platform, which suggest a fast yet volatile growth of A_t.

Because P_t converges to a multiple of A_t, μ_t^P converges to μ^A. Under the risk-neutral measure, the expected token price change has to be lower than r (otherwise agents invest as much as they can in tokens), so we set $\mu^A = 2\%$ and $r = 5\%$. The gap between μ^A and r determines how widely μ_t^P varies, which will be explained in detail with Figure 4, so we use the volatility of the percentage change of user addresses to discipline the choice of μ^A. Specifically, we map the data moment to the average σ_t^N/N_t in the states where $N_t \in [\underline{N}, 0.5]$.

After pining down the range and time span of N_t's growth, we set $\theta = 10/\sqrt{2}$ (i.e., a cross-section variance of u_i is 50), so that the curvature of N_t's path matches data. u_i's dispersion determines how responsive N_t is to the variation of A_t. The model generates a S-Shaped growth of N_t, consistent with the rising growth rate of user addresses in our sample.

α governs the co-movement between N_t and P_t. We can think of token as an asset that pays a flow dividend (trade surplus). α determines the decreasing return to N_t in the user-surplus flow. Therefore, a higher value of α increases the co-movement between P_t and N_t through a cash flow channel. We set α to 0.3 so that the model generates the comovement

[17] We start with $g(u) = \sqrt{\frac{1}{2\pi\theta^2}} e^{-\frac{u^2}{2\theta^2}}$ but truncate it at six-sigma tails and renormalized it.

Table 1: Calibration

	Parameter	*Value*	*Model*	*Data*
		Panel A: Key Parameters		
(1)	σ^A	200%	Growth rate of N_t	Growth of user address
(2)	θ	$10/\sqrt{2}$	Curvature of N_t growth	Curvature of user-address curve
(3)	α	0.3	Comovement: N_t & P_t	User address & crypto market cap
(4)	μ^A	2%	$\frac{\sigma_t^N}{N_t}$, vol. of N_t % change	Vol. of user-address % change
		Panel B: Other Parameters		
(5)	ϕ	1	Scaling effect on A_t	
(6)	M	10 billion	Monetary Neutrality	
(7)	ρ	1	Shock correlation: SDF & A_t	
(8)	r	5%	Risk-free rate	
(9)	η	1	Price of risk	

of P_t and N_t in the states where $N_t \in [\underline{N}, 0.5]$. In Appendix B, we perturb the values of four key parameters, μ^A, σ^A, θ, and α, to examine the robustness of our results, and through this comparative statics analysis, to confirm the intuition behind our parameter choice.

The remaining parameters do not affect much the equilibrium dynamics. Recall that ϕ is the participation cost, measured in goods. We set ϕ equal to 1 as a reference point for other parameters. We set M to 10 billion. Our model features monetary neutrality – the equilibrium dynamics stays the same, for instance, if M is doubled (so P_t is halved).

5 User Adoption and the Roles of Tokens

In this section, we show the results on the adoption dynamics and highlight the roles of tokens by comparing three economies, the tokenized economy that we have solved previously, a tokenless economy, and the first-best economy (i.e., the planner's solution). First, we set up the two benchmark economies, the tokenless and first-best.

5.1 Benchmark economies

Tokenless Economy. Suppose agents can conduct businesses and enjoy the trade surplus on the same blockchain platform using dollar, the numeraire, as the medium of exchange. The only difference compared to the tokenized economy is that agents' profits are no longer exposed to the token price fluctuation, μ_t^P:

$$dy_{i,t} = \max\left\{0, \max_{x_{i,t}>0}\left[\underbrace{(x_{i,t})^{1-\alpha}\left(N_t A_t e^{u_i}\right)^{\alpha} dt}_{\text{blockchain trade surplus}} - \underbrace{\phi dt}_{\text{participation cost}} - \underbrace{x_{i,t} r dt}_{\text{financing cost}}\right]\right\}. \tag{21}$$

Conditional on joining the platform (i.e., $x_{i,t} > 0$), the first order condition for $x_{i,t}$ yields

$$x_{i,t}^* = N_t A_t e^{u_i}\left(\frac{1-\alpha}{r}\right)^{\frac{1}{\alpha}} \tag{22}$$

The highest profit when joining the platform is then

$$N_t A_t e^{u_i}\alpha\left(\frac{1-\alpha}{r}\right)^{\frac{1-\alpha}{\alpha}} - \phi. \tag{23}$$

An agent joins the platform only when (23) is positive. Taking logarithm, we obtain the adoption threshold,

$$\underline{u}_t^{NT} = -\ln(N_t) + \ln\left(\frac{\phi}{A_t\alpha}\right) - \left(\frac{1-\alpha}{\alpha}\right)\ln\left(\frac{1-\alpha}{r}\right), \tag{24}$$

where the superscript "NT" is for "no tokens", so the user base is given by ,

$$N_t^{NT} = 1 - G\left(\underline{u}_t^{NT}\right). \tag{25}$$

Equations (24) and (25) jointly determine $\underline{u}_t^{NT}$ and N_t^{NT}. N_t^{NT} is increasing in A_t. We define $\underline{A}^{Tokenless}$ as $\underline{A}\left(\mu_t^P\right)$ in Proposition (1) with $\mu_t^P = 0$ so that $N_t > 0$ only if $A_t \geq \underline{A}^{Tokenless}$ in the tokenless economy.

First-best Economy From the tokenized economy to the tokenless economy, we strip off the tokens as medium of exchange. Next, we centralize agents' decision making and consider a planner's problem. Given a user base N_t, the socially optimal holdings of dollars is

$$x_{i,t}^* = N_t A_t e^{u_i}\left(\frac{1-\alpha}{r}\right)^{\frac{1}{\alpha}}. \tag{26}$$

Let $\mathcal{U}_t$ denote the set of user base with measure N_t. Then the total trade surplus (if positive) with user base N_t is

$$\int_{i\in\mathcal{U}_t}\left[\alpha N_t A_t e^{u_i}\left(\frac{1-\alpha}{r}\right)^{\frac{1-\alpha}{\alpha}}-\phi\right]di = N_t\left[\alpha\left(\frac{1-\alpha}{r}\right)^{\frac{1-\alpha}{\alpha}}A_t\int_{i\in\mathcal{U}_t}e^{u_i}di-\phi\right]. \tag{27}$$

To maximize this welfare flow, the planner should set $N_t = 1$, i.e., $\mathcal{U}_t$ being the full set of agents. However, given $N_t = 1$, if the planner's objective (27) is negative, then zero adoption would be socially optimal and the total welfare is zero. The switching from zero adoption to full adoption happens at

$$\underline{A}^{FB} = \phi\left[\alpha\left(\frac{1-\alpha}{r}\right)^{\frac{1-\alpha}{\alpha}}\int_{u=\underline{U}}^{\overline{U}}e^u g(u)du\right]^{-1}. \tag{28}$$

When $\int_{u=\underline{U}}^{\overline{U}}e^u g(u)du < \infty$, welfare maximization has a bang-bang solution, requiring full-adoption if $A \geq \underline{A}^{FB}$ and zero adoption otherwise.

It is straight forward to check that $\underline{A}^{FB} \leq \underline{A}^{Tokenless}$. Because $A_t = \underline{A}^{FB}$, the *average* maximized profit is larger than the participation cost, so it cannot be the case that for any agent with $u_i \geq \underline{u}_t^{NT}$, it is profitable to join. Therefore, there is under-adoption in a tokenless economy relative to the first-best level.

5.2 The Roles of Tokens in User Adoption

Next we illustrate the adoption acceleration and user-base volatility reduction effects of tokens with the numerical solutions.

Token acceleration effect. When token is introduced as the required medium of exchange on a platform, its market price reflects agents' anticipation of future technological progress and user adoption, which translates into the expected token price appreciation. Tokens therefore accelerates adoption because agents join the community not only to enjoy the trade surplus but also the return from rising token price.

The solid line in Figure 3 plots the user base N_t against the logarithm of A_t. The curve exhibits S-shaped development. When the platform is not so productive, the user base is not very responsive to A_t. But as N_t increases, the growth of user base feeds on itself – the

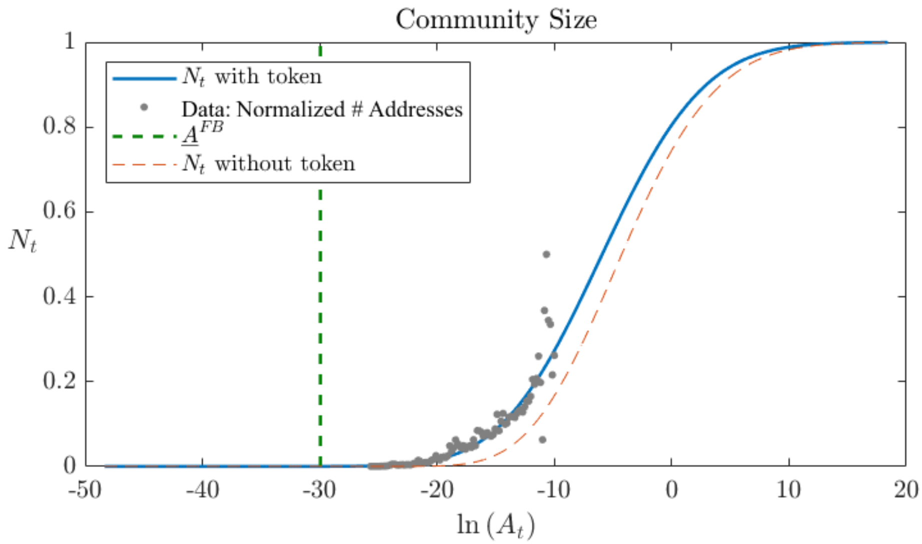

Figure 3: Dependence of User Base on Blockchain Productivity

more agents join the ecosystem, the higher surplus it is from transacting on the blockchain. As a result, the growth of N_t speeds up in the interim range of platform productivity. User adoption eventually slows down when the pool of newcomers get exhausted.

Both the growth rate and curvature of N_t over time match well the pattern in data. As previously discussed, we map the highest number of user addresses (December 2017) to $N_t = 0.5$, and record its corresponding value of $\ln(A_t)$ in our model. We scale the number of addresses in other months by that of December 2017. With December 2017 as the reference point, we calculate the corresponding value of $\ln(A_t)$ for each month by applying A_t's annual growth rate of 202% to December 2017. The leftmost state has $N_t = \underline{N} = 0.0001$.

Figure 3 also compares the user adoption in tokenized and tokenless (decentralized) economies. The former strictly dominates the latter. The two eventually converge to one as A_t grows. Notice that (11) differs from (24) by having the extra μ_t^P term. When $\mu_t^P > 0$, $N_t > N_t^{NT}$, where N_t is given in (12). In other words, expected token price appreciation induces a higher level of adoption than the case without tokens.[18] Token price appreciation is ultimately driven by the growth of A_t. When A_t (and thus, the user base and token demand) is expected to grow over time, agents forecast a higher token price, and the investment motive induces agents to Therefore, tokens accelerate user adoption when A_t a In a system without

[18]Note that in the system without token, transactions are settled on dollars, and we simplify the analysis by assuming that the price of dollar in goods is fixed at one. In reality, the value of dollar declines over time due to inflation, which strengthens the adoption acceleration effect.

tokens, the investment-driven demand is shut down.

Intuitively, if A_t is expected to grow fast (for example, due to a larger and positive μ^A), μ_t^P tends to be positive. Therefore, by introducing a native token that is required to be the medium of exchange, a blockchain system can capitalize future productivity growth in token price, and thereby accelerate adoption. This is the growth effect of introducing tokens in promising platforms (large and positive μ^A).[19]

A dark side of using tokens is that when A_t is expected to deteriorate (e.g., due to $\mu^A < 0$), the associated token depreciation ($\mu_t^P < 0$) precipitates the collapse of user base and the demise of the platform. Our analysis focuses on the case where $\mu^A > 0$.

We calculate $\underline{A}^{FB}$ using Equation (28), and show it by the dotted vertical line. According to our analysis, the planner chooses full adoption if $A_t \geq \underline{A}^{FB}$ and zero adoption if $A_t < \underline{A}^{FB}$. Due to agents' investment motive, tokenized economy may feature *over* adoption in the sense that $N_t > 0$ even if $A_t < \underline{A}^{FB}$. over adoption causes wasteful investment, particularly in the form of participation costs paid by users. Under the current calibration, over adoption does not seem to be a severe problem because N_t is extremely close to zero for $A_t < \underline{A}^{FB}$.

Token volatility reduction effect. Next, we compare the user base volatility in tokenized and tokenless economies. Note that in the first-best economy, since the adoption is either zero or full, user base volatility is not an issue.

Without native tokens, agents' decision to participate is purely driven by the current level of blockchain productivity A_t and their idiosyncratic transaction needs u_i. Therefore, the user base N_t varies only with A_t. Introducing tokens also changes the volatility of N_t through the endogenous dynamics of token price, because agents' decision to participate also depends on μ_t^P.

To derive the dynamics of N_t, we first conjecture that N_t follows a diffusion process in equilibrium

$$dN_t = \mu_t^N dt + \sigma_t^N dZ_t^A. \tag{29}$$

Strictly speaking, N_t follows a reflected (or "regulated") diffusion process that is bounded

[19] We note that a predetermined token supply schedule is important. If token supply can arbitrarily increase ex post, then the expected token price appreciation is delinked from the technological progress. Predeterminacy or commitment can only be credibly achieved through the decentralized consensus mechanism empowered by the blockchain technology. In contrast, traditional monetary policy has commitment problem – monetary authority cannot commit not to supply more money when its currency value is relatively high.

below at zero and bounded above at one, so we study the interior behavior of N_t. In the appendix, we show that in an economy without tokens, the diffusion of dN_t is

$$\sigma_t^N = \left(\frac{g\left(\underline{u}_t^{NT}\right)}{1 - g\left(\underline{u}_t^{NT}\right)/N_t^{NT}} \right) \sigma^A. \tag{30}$$

Meanwhile, in an economy with tokens, the diffusion of dN_t is

$$\sigma_t^N = \left(\frac{g\left(\underline{u}_t\right)}{1 - g\left(\underline{u}_t\right)/N_t} \right) \left[\sigma^A + \left(\frac{1-\alpha}{\alpha} \right) \left(\frac{\sigma_t^{\mu^P}}{r - \mu_t^P} \right) \right], \tag{31}$$

where, $\sigma_t^{\mu^P}$ is the diffusion of μ_t^P in its law of motion,

$$d\mu_t^P = \mu_t^{\mu^P} dt + \sigma_t^{\mu^P} dZ_t^A. \tag{32}$$

Comparing Equations (30) and (31), we see that introducing tokens could alter the volatility dynamics of user base through the fluctuation of expected token price change, i.e., $\sigma_t^{\mu^P}$. A priori, having a native token may either amplify or dampen the shock effect on the the user base, depending on the sign of $\sigma_t^{\mu^P}$. By Itô lemma, $\sigma_t^{\mu^P} = \frac{d\mu_t^P}{dA_t}\sigma^A A_t$, so the sign of $\sigma_t^{\mu^P}$ depends on whether μ_t^P increases or decreases in A_t.

Intuitively, μ_t^P decreases in A_t (and thus, $\sigma_t^{\mu^P} < 0$), precisely because of the endogenous user adoption. Consider an increase in A_t, which corresponds to an increase in N_t, reducing potential newcomers to join the community in future. Recall that token price appreciation is driven by the future increase in both A_t and N_t, so when there is less potential for N_t to grow, the expected token price appreciation, i.e., μ_t^P, declines. Because μ_t^P is decreasing in A_t, introducing token can *reduce* the conditional volatility of user base. That said, because $\underline{u}_t^{NT}$ and $\underline{u}_t$ could differ in general, there could be regions where the conditional volatility of user base is higher when token is introduced. In our calibration, we find that the region of intermediate adoption whereby tokens reduce user base volatility is significant. Later we numerically show that for most adoption levels, tokens reduce the user-base volatility.

Next, we numerically illustrate how tokens reduce user-base volatility. The left panel of Figure 4 plots σ_t^N, and compares the cases with and without token across different stages of adoption. Both curves starts and ends at zero, consistent with the S-shaped development in

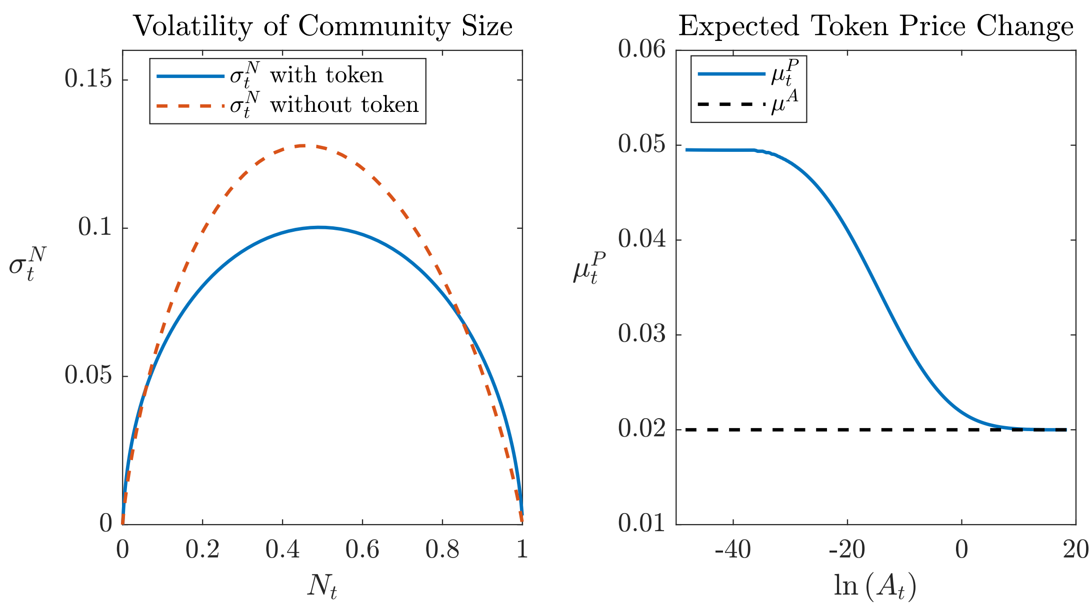

Figure 4: Volatility of User Base.

Figure 3 where both curves starts flat and ends flat. This volatility reduction effect is more prominent in the early stage of development when A_t and N_t are low. Note that σ_t^N can be slightly higher when token is introduced because the first brackets in Equations (30) and (31) differ due to the difference between $\underline{u}_t^{NT}$ and $\underline{u}_t$ even for the same adoption level.

The right panel of Figure 4 plots μ_t^P against $\ln(A_t)$, showing their negative relation that causes $\sigma_t^{\mu^P} < 0$, which generates the volatility reduction effect. When A_t is low and N_t is low, token price is expected to increase fast reflecting both the future growth of A_t and N_t. As A_t and N_t grow, the pool of newcomers is getting exhausted and there is less potential for N_t to growth. As a result, the expected token appreciation declines. Consider a positive shock to A_t (i.e., $dA_t > 0$), this negative impact on N_t through the expected token appreciation counteracts the direct positive impact on N_t through a higher trade surplus, making N_t less responsive to dA_t than the case without tokens. Similarly, following a negative shock to A_t, the trade surplus declines, but μ_t^P increases, which induces more agents to hold tokens than the case without tokens.

Remark: *Given the roles of the tokens in accelerating and stabilizing user adoption, entrepreneurs may want to introduce them in a platform. For example, suppose the platform can collect a fee of ϕ from the users, greater adoption would increase the revenue of the platform. One way to kill two birds with one stone is to issue tokens to early investors through*

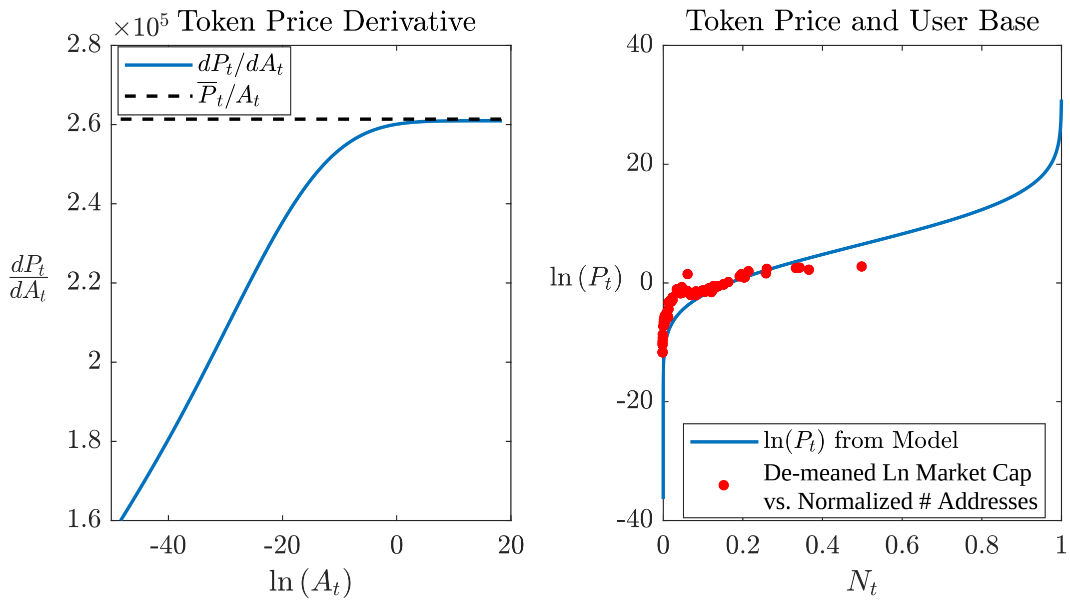

Figure 5: Dependence of Token Price on Blockchain Productivity and User Base.

ICOs which brings in capital for developing the platform as well. Then through retaining some tokens, the early investors and entrepreneurs can also enjoy the token price appreciation. Our on-going work and Bakos and Halaburda (2018) explore such considerations of the entrepreneurs and platforms designers.

6 Endogenous Adoption on Token Price

Token Price. The token price is solved as a function of blockchain productivity A_t. The left panel of Figure 5 plots dP_t/dA_t against $\ln(A_t)$. The curve starts at $\ln(A_t) = -48.35$ ($A_t = 1e - 21$), a number that we choose to be close to zero, the left boundary. The curve ends at $\ln(A_t) = 18.42$ ($A_t = 1e8$), the touching point between $P(A_t)$ and $\overline{P}(A_t)$. A key message from the left panel is that over time, token price becomes increasingly sensitive to the variation in A_t. When the user base is small, token price is less responsive to the growth of A_t, because A_t is multiplied by N_t when entering into the blockchain trade surplus. As A_t grows and N_t approaches 1, P_t becomes more sensitive to A_t.

The right panel of Figure 5 plots the logarithm of token price against the size of user base, both functions of A_t in equilibrium. On a logarithm scale, token price increases fast with adoption in the early stage, and then gradually rises with user base, but when the

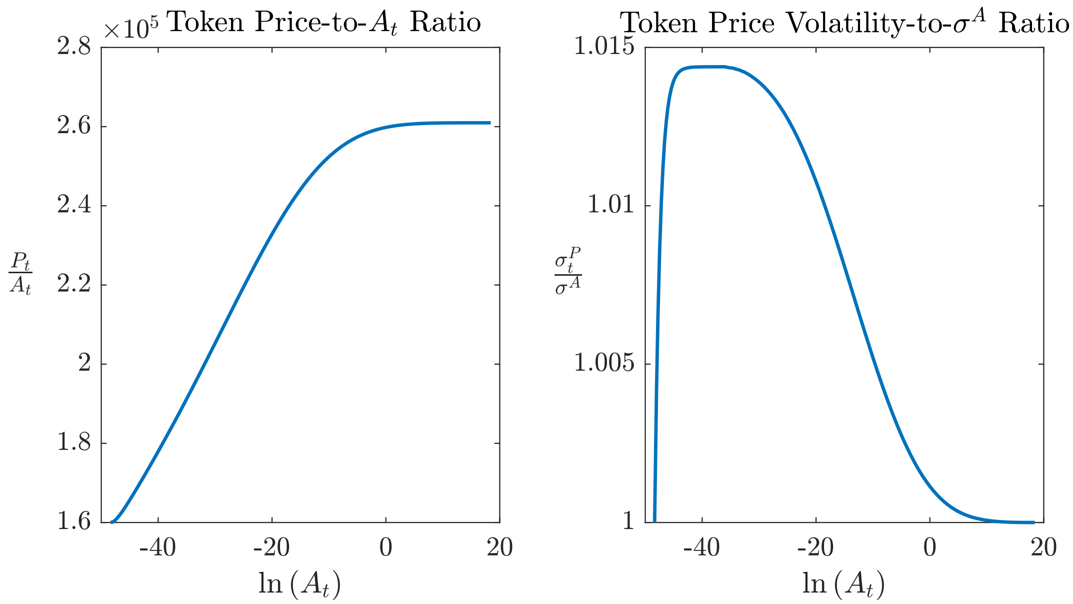

Figure 6: Token Price-to-Blockchain Productivity Ratio and Volatility.

system has accumulated a critical mass of users, the increase of token price speeds up and converges to its long-run path. More generally, token price could have exponential growth with respect to the user base, an important implication for trading cryptocurrencies. The comovement of P_t and N_t matches well the pattern in data, for which we only observe the early stage of adoption.

The Run-up of Token Price Volatility. The dynamics of user adoption in turn affects the volatility of token price. When $\phi = 0$, agents' decision to participate becomes irrelevant. Every agent participates, so $N_t = 1$, and token price is given by Equation (17) and the ratio of P_t to A_t is a constant. Therefore, the diffusion of token price, i.e., σ_t^P, is equal to σ^A.

A key theme of this paper is the endogenous dynamics of user adoption. When $\phi > 0$, the ratio of P_t to A_t depends on the $\underline{u}_t$, the threshold value of agent-specific needs for blockchain transactions, above which an agent participates. The variation of N_t feeds into P_t/A_t, and therefore, amplifies the volatility of token price beyond σ^A, the level of volatility when the issue of user adoption is irrelevant.

The left panel of Figure 6 plots P_t/A_t. As shown in Equation (14), this ratio follows closely the dynamics of N_t shown in Figure 3, but is steeper in the early stage (i.e., low A_t region). The right panel of Figure 6 plots the ratio of token price volatility to σ^A, which eventually converges to 1 as N_t approaches one and P_t approaches its asymptote. At its height, endogenous user adoption amplifies token price volatility (or instantaneous standard deviation, to be precise) by 1.4%, but the excess volatility eventually declines with further adoption. We remind the reader that the result obtains under the premise that token price is purely driven by the fundamental productivity and there is no token/platform competition as described in Section 8. A larger volatility can be observed in practice because of platform competition and non-fundamental-based speculations.

What is interesting is the qualitative implications: as a new platform is gradually adopted, one may observe dormant token price variation, followed by a volatility run-up before the eventual stabilization. A key result of this paper is this the mutual effect between N_t and P_t in both growth and volatility. Especially in the early stage of adoption, user participation amplifies the growth and volatility of token price, while at the same time, is being affected by agents' expectation of future token price appreciation and the volatility of such expectation.

Risk and Return under the Physical Measure The expected token price appreciation under the physical measure is

$$\hat{\mu}_t^P = \mu_t^P + \eta\rho\sigma_t^P. \tag{33}$$

The covariance between token price change and SDF shock, i.e., $\rho\sigma_t^P$, is priced at η. If the shock to blockchain productivity is orthogonal to SDF shock ($\rho = 0$), then $\hat{\mu}_t^P = \mu_t^P$. Next, we graphically illustrate how risk premium varies across different stages of platform development. Figure 7 plots the token risk premium, i.e., $\hat{\mu}_t^P - r$, under the physical measure against $\ln(A_t)$. As A_t increases, the risk-neutral drift of token price (μ_t^P) declines, while the volatility (σ_t^P) follows a hump-shaped curve, which dominates the dynamics of risk premium.

Our risk premium of 200% is higher than the average annual return to the cryptocurrency portfolio in our sample (27%). The main reason is the decline of cryptocurrency market value in 2018. So far, we have fixed the correlation between SDF shock and shock to A_t as a constant. Yet as a blockchain platform or the general technology gains popularity, eases, its token is becoming a systematic asset. Pástor and Veronesi (2009) emphasize that the beta of new technology tends to increase as it becomes mainstream and well adopted. We can allow the correlation between SDF and A_t to depend on N_t by decomposing the technological

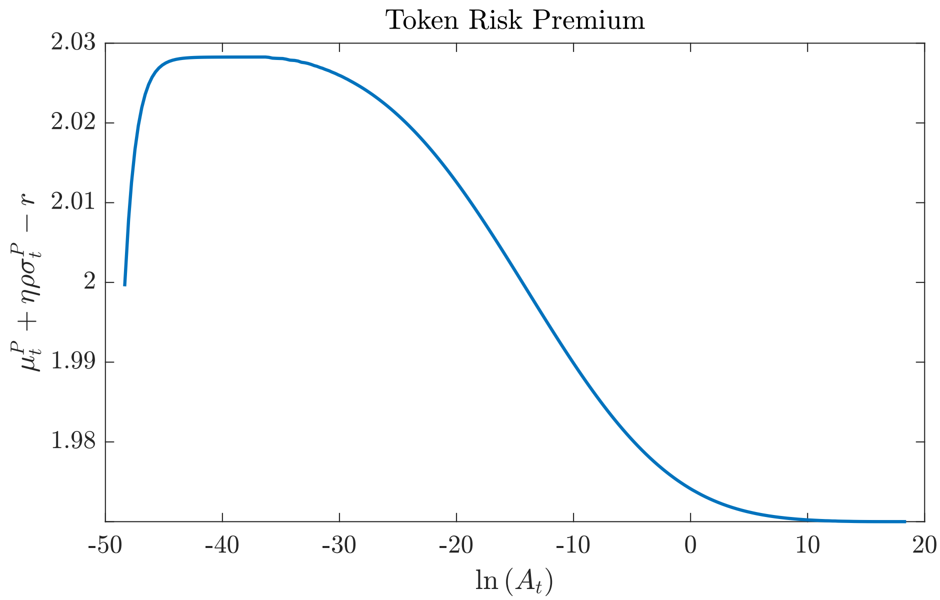

Figure 7: Expected Capital Gain under the Physical Measure.

shock into two components under the physical measure,

$$d\hat{Z}_t = \rho(N_t)\, d\hat{Z}_t^{\Lambda} + \sqrt{1-\rho(N_t)^2}\, d\hat{Z}_t^{I}, \tag{34}$$

where the standard Brownian shock, $d\hat{Z}^I$, is independent from the SDF shock, $d\hat{Z}_t^{\Lambda}$. Therefore, the correlation between technological shock and SDF shock is $\rho(N_t)$, where we assume $d\rho/dN_t > 0$, that is the blockchain productivity shock becomes increasingly systematic as the user base grows. Under the risk-neutral measure, we have

$$\frac{dA_t}{A_t} = \left[\hat{\mu}^A - \eta\rho(N_t)\,\sigma^A\right] dt + \sigma^A dZ_t, \tag{35}$$

so the risk-neutral, expected growth rate of A_t is $\hat{\mu}^A - \eta\rho(N_t)\,\sigma^A$, which now declines in N_t.

Therefore, as A_t grows, there are two opposing forces that drive P_t. On the one hand, the mechanisms that increase P_t are still there: when A_t directly increases the flow utility of token, or indirectly through N_t, token price increases. On the other hand, through the increase of N_t, the expected growth of A_t under the *risk-neutral* measure declines, which pushes P_t down. If our previous mechanisms work in the early stage of adoption while the channel of N_t-dependent token beta dominates in the later stage of adoption, what we shall see in the equilibrium will be a bubble-like behavior – P_t rises initially, and later as N_t rises,

P_t declines because the risk-neutral expectation of A_t growth declines.

7 Institutional Background

In this section, we describe the development of the blockchain technology, and then clarify various concepts associated with cryptocurrency, which are not mutually exclusive and are starting to be used interchangeably. Importantly, we highlight two salient features shared among the majority of cryptocurrencies, crypto-tokens, and platform currencies: first, they are used as media of exchange by design ("token embedding"); second, their typical application scenarios exhibit some forms of network effect ("user-base externality").

Blockchain, Cryptocurrency, and Token. The advances in FinTech and sharing economy is largely driven by the increasing preference for forming peer-to-peer connections that are instantaneous and open, which is transforming how people interact, work together, consume and produce. Blockchain-based applications are part of an attempt to create a financial architecture to reorganize the society into a set of networks of human interactions, allowing peers unknown to and distant from one another to interact, transact, and contract without relying a centralized trusted third party. The technology is believed to potentially avoid single point of failure and even reduce concentration of market power, but still face many challenging issues.[20]

Even though not always necessarily required, a majority of blockchain applications entail the use of cryptocurrencies and tokens. Cryptocurrencies are cryptography-secured digital or virtual currencies. Bitcoin represents the first widely-adopted decentralized cryptocurrency, and popularized the concept.[21] Besides Bitcoin, over 1000 different "altcoins" (stand for for alternative cryptocurrency coins, alternative to Bitcoin) have been introduced over the past few years and many central banks are actively exploring the area for retail and payment

[20]Although Bank of England governor Mark Carney dismissed Bitcoin as an alternative currency, he recognized that the blockchain technology benefits data management by improving resilience by "eliminating central points of failure" and enhancing transparency and auditability while expanding what he called the use of "straight-through processes" including with smart contracts. In particular, "Crypto-assets help point the way to the future of money". See, e.g., beat.10ztalk.com. For various applications of the technology, we refer the readers to Harvey (2016) and Yermack (2017), and for smart contracting, Cong and He (2018).

[21]Many retailers in Japan already accept Bitcoins (e.g., Holden and Subrahmanyam (2017)), not to mention that many ICOs are paid using Bitcoins.

systems.[22] Many altcoins such as Litecoin and Dogecoin are variants (forks) from Bitcoin, with mofidications to the original open-sourced protocol to enable new features. Others such as Ethereum and Ripple created their own Blockchain and protocol to support the native currency. Cryptocurrencies are typically regarded as payment-focused and primarily associated with their own independent blockchain. In these payment and settlement applications as exemplified by Bitcoin and Ripple, cryptocurrencies obviously act as media of exchange on their respective blockchain platforms.

Meanwhile, Blockchain-based crypto-tokens have also gained popularity. In what is known as Initial Coin Offerings (ICO), entrepreneurs sell "tokens" or "AppCoins" to dispersed investors around the globe.[23] Tokens are representations of claims on issuers' cashflow, rights to redeem issuers' products and services, or media of exchange among blockchain users. They usually operate on top of an existing blockchain infrastructure to facilitates the creation of decentralized applications.[24]

However, it is far from clear how cryptocurrencies and tokens derive values and how they should be adopted given their large volatilities. Consequently, there lacks no critics of the development of cryptocurrency, at least Bitcoin, in both the industry and academia.[25] ICOs are also facing quagmires regarding its legitimacy and distinction from security issuance.[26] In the recent hearing on Capital Markets, Securities, and Investment Wednesday, March 14, 2018, the regulators also appear rather divided on the future of cryptocurrencies, digital currencies, ICOs, and Blockchain development.

Obviously, some tokens derive their value from the company's future cashflow, and thus,

[22]For example, People's Bank of China aims to develop a digital currency system; Bank of Canada and Monetary Authority of Singapore use blockchain for interbank payment systems; Deutsche Bundesbank works on prototype of blockchain-based settlement systems for financial assets; in a controversial move, the government of Venezuela became the first federal government to issue digital currency and announced on Feb 20, 2018 the presale of its "petro" cryptocurrency — an oil-backed token as a form of legal tender that can be used to pay taxes, fees and other public needs.

[23]While the first ICO in 2013 raised a meager $ 500k and sporadic activities over the next two years. 2016 saw 46 ICOs raising about $ 100m and according to CoinSchedule, in 2017 there were 235 Initial Coin Offerings. The year-end totals came in over $3 billion raised in ICO. In August, 2017, OmiseGO (OMG) and Qtum passed a US$1 billion market cap today, according to coinmarketcap.com, to become the first ERC20 tokens built on the Ethereum network and sold via an ICO to reach the unicorn status.

[24]By "on top of" a blockchain, we mean that one can use smart contract templates, for example on the Ethereum or Waves platform, to create tokens for particular applications, without having to create or modify codes from other blockchain protocols.

[25]Rogoff (2017) and Shiller (2017) are notable representatives, although Cochrane argues otherwise in *The Grumpy Economist*, Nov 20, 2017.

[26]See, for example, "Token Resistance," The Economist, November 11th, 2017.

serve a function similar to securities (thus termed "security tokens"). The vast majority of ICOs that launched in 2016 and 2017 were "utility tokens", which include many of the highest-profile projects: Filecoin, Golem, 0x, Civic, Raiden, Basic Attention Token (BAT), and more. As we illustrate using some of the aforementioned tokens shortly, utility tokens are often the required media of exchange among users for certain products or services, or represent certain opportunities to provide blockchain services for profit as in the case of "stake tokens". Precisely here lies the key innovation of the blockchain technology: allowing peer-to-peer interactions in decentralized networks, as opposed to designing and auctioning coupons issued by centralized product/service providers – an old phenomenon economists understand relatively well (e.g., pay-in-kind crowdfunding).[27]

In this paper, we focus on the common features shared by cryptocurrencies and utility tokens that serve essentially the role of media of exchange among blockchain users. We thus use "tokens", "cryptocurrencies", etc., interchangeably and often collectively refer to them as "tokens". Next, we highlight these unique features of the blockchain technology that distinguishes the economics of introducing and valuing tokens from what we already know in the literature of monetary economics and asset pricing.

Token Embedding. Many blockchain-based decentralized networks introduce native currencies – a phenomenon we call "token embedding". In the following, we elaborate on the rationales behind such phenomenon and relate them to the issue of money velocity, motivating our formal analysis in Section 2.

First, in the virtual economy, potential users are likely from around the globe, using fiat money issued by and subject to specific countries' legal and economic influences. Transacting in a uniform currency is simply more convenient, free from the transaction costs of currency exchange. For example, it is cheaper to make international payments and settlements using Ripples (XRP) on the Ripple network. Even though Ethereum platform allows other AppCoins and cryptocurrencies (provided that they are ERC-20 compatible), many transactions and fundraising activities are still carried out using Ethers (ETH) because of its convenience and popularity (i.e., widely accepted by Ethereum users).

[27] Media often analogizes utility tokens to "corporate coupons", which allow consumers to redeem products or services from the service provider. Although some tokens are indeed corporate coupons, it is thus far neglected that the majority of them are not. Not only the valuation framework differs, but the legal and regulatory implications differ as well.

Second, from a theoretical perspective, it is advantageous to adopt a standard unit of account in the ecosystem because it mitigates the risks of asset-liability mismatches due to denomination in different units of account (Doepke and Schneider (2017)). This is particularly relevant on a blockchain platform designed for smart contracting.

Then why not just use US dollars or other existing currencies as settlement media? This leads to the third rationale: native currency helps to incentivize miners, validators, and users to contribute to the stability, functionality (provision of decentralized consensus), and prosperity of the ecosystem (Nakamoto (2008)). For example, for blockchain applications where decentralized consensus is achieved through the mechanism of "proof-of-stake", the ownership of native currency entitles platform users to be the consensus generator/recorder; for blockchains relying on "proof-of-work" such as Bitcoins and Filecoins, native tokens are used to reward miners for block creations in the consensus processes; moreover, to profit from providing validation services, OmiseGo tokens (OMG) are required as proof of stakes on the OmiseGo blockchain. If a blockchain application is developed without a native currency, then the incentive of users is no longer directly linked to the platform in question. Practitioners are very well aware of this issue, as Strategic Coin explains in its BAT token launch research report.[28]

Fourth, introducing native currency allows the issuer to collect seigniorage, especially through ICOs (e.g., Canidio (2018)). In contrast to sovereigns who cannot easily commit to a money supply rule, blockchain developers can commit to an algorithmic rule of token supply to generate scarcity. Provided that users need to hold tokens to transact on the platform, a positive token price can arise in an equilibrium, and such value is collected by the developers at ICO, reflecting a form of monopoly rent – the fact that users can only conduct some activities on a particular blockchain platform translates into a high price of its native currency, and more ICO revenues to the developer.

These rationales motivate us to focus on platforms with native tokens. But we still need to ask why cryptocurrency may have a determinant value in the first place. In principle, if one wants to transact on a blockchain platform, one can exchange dollars for its native currency, and make a transfer on the blockchain, and then immediately, the payee may exchange the

[28]BAT serves as a medium exchange between users, advertisers, and publishers who participate in the Brave browser ecosystem. Advertisers purchase ads using BAT tokens, which are then distributed among both publishers and browser users as compensation for hosting the ads and viewing them, respectively.

native currency back into dollars. If the whole process happens instantaneously, i.e., the velocity of native currency is infinite, then there does not exist a net demand for native currency, so there exists an equilibrium of zero dollar price and equilibria with any positive level of price of native currency. Therefore, we need to pin down a positive demand, so that with a algorithmically controlled supply, token price can be determined. This brings us to the second aspect of token embedding: agents actually need to hold the medium of exchange to profit from on-chain activities. This is indeed the case in practice for at least three reasons, in addition to the obvious practical concern that converting between the native token and other fiat money have high fixed costs—a concept similar to convenience yield of currencies and commodities.

First, a demand may arise because decentralized miners or service providers ("keepers") may have to hold the native currencies to earn the right to serve the system. Proof-of-Stake protocols typically fall in this category. These tokens are sometimes referred to as work tokens or staking tokens, and notable implementations include Keep (off-chain private computation), Filecoin (distributed file storage), Truebit (off-chain computation), Livepeer (distributed video encoding), and Gems (decentralized mechanical Turk). To enforce some sort of mechanism to penalize workers who fail to perform their job to some pre-specified standard, work tokens have to be held. For example, in Filecoin, service providers contractually commit to storing some data with 24/7 access and some minimum bandwidth guarantee for a specified period of time. During the contract term, service providers must "escrow" some number of Filecoin, which can be automatically slashed (taken away) should they fail to perform the service. Staking tokens in OmiseGo are also emerging, where locking up tokens in a smart contract allows a user to access the market place.

Second, blockchains enable the use of smart contracts (Cong and He (2018)). Though not yet widely implemented, smart contracts may involve automated transfers for contingencies specified over an extended period of time, effectively requiring escrowing the tokens.[29] In other words, agents hold cryptocurrency as collateral. While this is similar to the traditional third-party escrow accounts, what it implies is that the tokens are locked up with at least one contracting party.

[29] Balvers and McDonald (2017) also argues that automated collateral in terms of tokens can help stabilize the purchasing power of cryptocurrency, a point very related to our emphasize on a positive crytocurrency demand.

Third, there are technical and legal limits on how quickly transactions can be validated and accepted. While many protocols such as the Lightening Network and Ethereum process transactions significantly faster than Bitcoin (seconds versus 10-11 minutes), the decentralized nature of the validation means it always takes some amount of time to ensure robustness and synchronization of the consensus.[30] During such confirmation time, cryptocurrency cannot be liquidated by either party of the transfer. For example, a coinbase account requires at least three confirmations—blocks added subsequently—before a transaction shows up and become spendable. Related is the traditional anti-money-laundering practice where one party of the transaction needs to show the funding source and prove it has been in the account for a certain period of time, as seen for example in the KYC (know-your-customer) process for investing in the pre-sale of Dfinity tokens.

Next, we discuss user-base externality, another key feature of decentralized network that our model captures.

User-Base Externality. User-base externality has been well recognized as one of the defining features of P2P platforms, sharing economy, and various decentralized systems. When more people join the platform, individuals enjoy more surplus through interacting with other users because it is easier to find a trade partner. "Trade" here can be very general, encompassing selling products and services and signing a long-term financing contracts. The utility of using cryptocurrencies and crypto-tokens obviously goes up when more people use the blockchain platform.[31] Moreover, UnikoinGold on Unikrn (decentralized token for betting on e-sports and gambling) and Augur (decentralized prediction market) are examples showing that achieving a critical mass is crucial in platform business (e.g., Evans and Schmalensee (2010)).

That said, existing discussions on user-base externality are often static, leaving out intertemporal effects that can be even more important. The fact that a larger user base today helps improve the technology tomorrow, and a larger anticipated user base tomorrow en-

[30] Chiu and Koeppl (2017) show theoretically that delaying settlement helps overcome the "double-spending problem", which refers to agents paying the same token to multiple counterparties.

[31] According to CoinSchedule, 34.5% of the ICO-financed projects over the past two years focus on infrastructure. While the other top categories included trading and investing at 13.7%, finance at 10.2%, payments at 7.8%, data storage also at 7.8%, and drugs and healthcare at 5.5%, amongst dozens of other industry categories. Regardless the ICO category, these projects share user-base externality as a common attribute, and in terms of user adoption, exhibit a S-shaped development curve – the growth of user base feeds on itself.

courages greater investments today are examples of how user-base externality can play an inter-temporal role. Filecoin the data storage network, Dfinity the decentralized computation infrastructure, marketplace such as overstock (and its ICO), and infrastructure projects such as Ethereum and LITEX all exhibit user-base externality in both contemporaneous and inter-temporal fashions, as our model highlight.

Commons and Assets with Network Effect. Given token embedding and user-base externality, tokens essentially constitute an asset that delivers owners "dividends" (in terms of user surplus) which increase in the scale of the platform. This type of assets are not restricted to tokens or blockchains. Network effect or user-base externality is prevalent and particularly important in the early stage of adoption for social networks and payment networks such as Facebook, Twitter, YouTube, WeChat, and PayPal. Other examples of such assets include membership of clubs with benefits growing in the network size, and collectibles of limited edition for sports teams.

As such, the insights gleaned from our model may apply to platform currencies in general, such as those used in interactive online games (e.g., World of Warcraft), virtual worlds (e.g., Second Life), and social networks (e.g., Facebook). Our findings also help inform what is likely to happen for other sharing economy applications such as UBER and AirBnB to introduce native tokens. In fact, consistent with our model, when Tencent QQ introduced Q-coin, a case to which our model is applicable, many users and merchants quickly started accepting them even outside the QQ platform (mapped to increase in system trade surplus in our model), tremendously accelerating adoption and token price appreciation.[32]

At a holistic level, this type of asset is important in the creation of commons that underly modern life. TCP/IP, HTTP, GPS, and the English language are some early examples.[33] Instead of having government or private firms facilitating the building of commons, blockchains offer new possibilities and as we show earlier, token embedding plays important roles in growing commons.

[32] Annual trading volume reached billions of RMB in late 2000s and the government has to intervene. See articles *China bars use of virtual money for trading in real goods* and *QQ: China's New Coin of the Realm?* (WSJ). Halaburda and Sarvary (2016) provide comprehensive discussions on various platform currencies.

[33] Alex Tabarrok gives a nice treatise on the concept in *Economics, Web/Tech* June 4, 2018, "Blockchains and the Opportunity of the Commons.

8 Conclusion

This paper provides the first dynamic pricing model of cryptocurrencies and tokens, taking into consideration the user-base externality and endogenous adoption. Our model highlights a key benefit of introducing cryptocurrencies and crypto-tokens on blockchain platforms: when agents expect future technology or productivity progress, token price appreciation induces more agents to join the platform by serving as an attractive investment. In other words, token capitalizes future growth and speeds up user adoption, which is often welfare-enhancing. Tokens can also reduce user-base volatility. We characterize the intertemporal feedback mechanism and show it leads to an S-shaped adoption curve in equilibrium, and token price dynamics crucially depend on the platform productivity, endogenous user adoption, and user heterogeneity.

Our model is flexible enough to admit multiple extensions, and we sketch four of them in Appendix C: 1. one can endogenize the productivity growth, which further strengthens the dynamic feedback channel we highlight; 2. one can allow time-varying systematic risk of tokens, which can produce a sharp rise and fall of token price under rational expectations; 3. cryptocurrency competition can be modeled by specifying reflecting boundary conditions; 4. one can introduce the design of state-contingent token supply for incentive provision for the supply of blockchain consensus. More generally, our framework can be applied to dynamic pricing of assets associated with a platform or system with network externality.

Appendix A - Proofs

A1. Proof of Proposition 1

Figure 1 illustrates the determination of N_t given A_t and μ_t^P, which we take as a snapshot of the dynamic equilibrium with time-varying productivity and expectation of price change. The proof below takes the following steps. First, we show that given μ_t^P, there exists a $\underline{A}$ such that for $A_t = A > \underline{A}$, the corresponding response curve,

$$R\left(n; A, \mu_t^P\right) = 1 - G\left(\underline{u}\left(n; A, \mu_t^P\right)\right) \tag{36}$$

$$= 1 - G\left(-\ln(n) + \ln\left(\frac{\phi}{A_t\alpha}\right) - \left(\frac{1-\alpha}{\alpha}\right)\ln\left(\frac{1-\alpha}{r-\mu_t^P}\right)\right), \tag{37}$$

crosses the 45^o line at least once in $(0,1]$, and for any value of $A_t = A < \underline{A}$, the response curve never crosses the 45^o line in $(0,1]$. After proving the existence of $N_t > 0$ for $A_t \in [\underline{A}, +\infty)$, we prove the uniqueness given the increasing hazard rate of $g(u)$. Finally, we prove that N_t increases in μ_t^P. Before we start, for any $A_t = A > 0$, we define the value of its response function at $n = 0$: $R\left(0; A, \mu_t^P\right) = 0$. This is consistent with that given a zero user base, each agent derives zero transaction surplus from token holdings and choose not to participate. Note that $\lim_{n\downarrow 0} R\left(n; A, \mu_t^P\right) = 0$, so the response function is continuous in n.

Given μ_t^P, we define a mapping, $A(n)$, from any equilibrium, non-zero value of user base, $n \in (0,1]$, to the corresponding value of A_t, i.e., the unique solution to

$$1 - G\left(-\ln(n) + \ln\left(\frac{\phi}{A_t\alpha}\right) - \left(\frac{1-\alpha}{\alpha}\right)\ln\left(\frac{1-\alpha}{r-\mu_t^P}\right)\right) = n, \quad \forall n \in (0,1]. \tag{38}$$

This mapping is a continuous mapping on a bounded domain $\subseteq (0,1]$. Then by the Least-Upper-Bound-Property of real numbers, the imagine set of this mapping, $\{A(n), n \in (0,1)\}$, has an infimum, which we denote by $\underline{A}$. Now, for $A_t = \underline{A}$, consider a $n\left(\underline{A}\right) \in (0,1]$ such that Equation (38) holds. For any $A > \underline{A}$, the LHS of Equation (38) is higher than the RHS, i.e., $R\left(n\left(\underline{A}\right); A, \mu_t^P\right) > n\left(\underline{A}\right)$, so that the response curve of $A_t = A$ is above the 45^o line at $n\left(\underline{A}\right)$. Next, because the response function $R\left(n; A, \mu_t^P\right)$ is continuous in n and $R\left(1; A, \mu_t^P\right) \leq 1$ by definition in Equation (37), i.e., it eventually falls to or below the 45^o line as n increases, there must exists a $n(A) \in (0,1]$ such that when at $A_t = A$, Equation (38) holds by the

Intermediate Value Theorem. Therefore, we have proved that for any $A_t = A > \underline{A}$, there exists a non-zero user base. Throughout the proof, we fix μ_t^P, so $\underline{A}$ is a function of μ_t^P.

Next, given $\frac{g(u)}{1-G(u)}$ is increasing, we show that the response curve crosses the 45^o line exactly once when $A_t \in [\underline{A}, +\infty)$. First note that $R\left(n; A_t, \mu_t^P\right) - n$ either has positive derivative or negative derivative at $n = 0$. If it has positive derivative (i.e., the response curve shoots over the 45^o line), then at n', *the first time* the response curve crosses the 45^o line again, the derivative of $R\left(n; A_t, \mu_t^P\right) - n$ must be weakly negative at n', i.e., the response curve crosses the 45^o from above,

$$g\left(\underline{u}\left(n'; A_t, \mu_t^P\right)\right) \frac{1}{n'} - 1 \leq 0. \tag{39}$$

Now suppose the response curve crosses the 45^o line for the second time from below at $n^{''} > n'$, so the derivative of $R\left(n; A_t, \mu_t^P\right) - n$ at n'' must be weakly positive, and is equal to

$$\begin{aligned} g\left(\underline{u}\left(n''; A_t, \mu_t^P\right)\right) \frac{1}{n''} - 1 &= \frac{g\left(\underline{u}\left(n''; A_t, \mu_t^P\right)\right)}{1 - G\left(\underline{u}\left(n''; A_t, \mu_t^P\right)\right)} - 1 \\ &< \frac{g\left(\underline{u}\left(n'; A_t, \mu_t^P\right)\right)}{1 - G\left(\underline{u}\left(n'; A_t, \mu_t^P\right)\right)} - 1 \\ &= \frac{g\left(\underline{u}\left(n'; A_t, \mu_t^P\right)\right)}{n'} - 1 \\ &< 0, \end{aligned} \tag{40}$$

where the first inequality comes from the increasing hazard rate and the fact that $\underline{u}\left(n; A_t, \mu_t^P\right)$ is decreasing in n for $n \in (0, 1]$, and the second inequality follows from (39) and the fact that the response curve crosses the 45^o line at n' (i.e., $n' = R\left(n'; A_t, \mu_t^P\right) = 1 - G\left(\underline{u}\left(n'; A_t, \mu_t^P\right)\right)$). This contradicts the presumption that the response curve reaches the 45^o line from below (and the derivative of $R\left(n; A_t, \mu_t^P\right) - n$ is weakly positive). Therefore, we conclude that for $A_t \in [\underline{A}, +\infty)$, there exists a unique adoption level n. Now if $R\left(n; A_t, \mu_t^P\right) - n$ has negative derivative at $n = 0$, then in the previous argument, we can replace n' with 0 and show that there does not exist another intersection between the response curve and the 45^o line beyond $n = 0$. Therefore, only if $R\left(n; A_t, \mu_t^P\right) - n$ has positive derivative at $n = 0$, do we have a positive (non-degenerate) adoption level.

Finally, we show that a non-degenerate adoption level, N_t, is increasing in μ_t^P. Consider

$\tilde{\mu}_t^P > \mu_t^P$. Suppose the contrary that their corresponding adoption levels satisfy $\tilde{N}_t \leq N_t$. Because we have proved that the response curve only crosses the 45^o line only once and from above, given N_t, we have

$$1 - G\left(-\ln(n) + \ln\left(\frac{\phi}{A_t\alpha}\right) - \left(\frac{1-\alpha}{\alpha}\right)\ln\left(\frac{1-\alpha}{r-\mu_t^P}\right)\right) \geq n, \quad \forall n \in (0, N_t]\,. \tag{41}$$

We know that by definition,

$$\begin{aligned}
\tilde{N}_t &= 1 - G\left(\underline{u}\left(\tilde{N}_t; A_t, \tilde{\mu}_t^P\right)\right) \\
&= 1 - G\left(-\ln\left(\tilde{N}_t\right) + \ln\left(\frac{\phi}{A_t\alpha}\right) - \left(\frac{1-\alpha}{\alpha}\right)\ln\left(\frac{1-\alpha}{r-\tilde{\mu}_t^P}\right)\right) \\
&> 1 - G\left(-\ln\left(\tilde{N}_t\right) + \ln\left(\frac{\phi}{A_t\alpha}\right) - \left(\frac{1-\alpha}{\alpha}\right)\ln\left(\frac{1-\alpha}{r-\mu_t^P}\right)\right) \\
&\geq \tilde{N}_t,
\end{aligned} \tag{42}$$

where the first inequality uses $\tilde{\mu}_t^P > \mu_t^P$ and the second inequality uses the fact that $\tilde{N}_t \in (0, N_t]$ and the inequality (41). This contradiction implies that the adoption level N_t has to be increasing in μ_t^P.

A2. Proof of Proposition 5.2

First, we consider the case without token. Using Itô's lemma, we can differentiate Equation (25), and then, by matching coefficients with Equation (29), we can derive the expressions for μ_t^N and σ_t^N:

$$dN_t = -g\left(\underline{u}_t^{NT}\right) d\underline{u}_t^{NT} - \frac{1}{2}g'\left(\underline{u}_t^{NT}\right)\left\langle d\underline{u}_t^{NT}, d\underline{u}_t^{NT}\right\rangle, \tag{43}$$

where $\left\langle d\underline{u}_t^{NT}, d\underline{u}_t^{NT}\right\rangle$ is the quadratic variation of $d\underline{u}_t^{NT}$. Using Itô's lemma, we differentiate Equation (24)

$$\begin{aligned} d\underline{u}_t^{NT} &= -\frac{1}{N_t}dN_t + \frac{1}{2N_t^2}\langle dN_t, dN_t\rangle - \frac{1}{A_t}dA_t + \frac{1}{2A_t^2}\langle dA_t, dA_t\rangle \\ &= -\left(\frac{\mu_t^N}{N_t} - \frac{\left(\sigma_t^N\right)^2}{2N_t^2} + \mu^A - \frac{\left(\sigma^A\right)^2}{2}\right)dt - \left(\frac{\sigma_t^N}{N_t} + \sigma^A\right)dZ_t^A \end{aligned} \tag{44}$$

Substituting this dynamics into Equation (43), we have

$$\begin{aligned} dN_t = &\left[g\left(\underline{u}_t^{NT}\right)\left(\frac{\mu_t^N}{N_t} - \frac{\left(\sigma_t^N\right)^2}{2N_t^2} + \mu^A - \frac{\left(\sigma^A\right)^2}{2}\right) - \frac{1}{2}g'\left(\underline{u}_t^{NT}\right)\left(\frac{\sigma_t^N}{N_t} + \sigma^A\right)^2\right]dt \\ &+ g\left(\underline{u}_t^{NT}\right)\left(\frac{\sigma_t^N}{N_t} + \sigma^A\right)dZ_t^A, \end{aligned} \tag{45}$$

By matching coefficients on dZ_t^A with Equation (29), we can solve σ_t^N.

Next, we consider the case with token. Once token is introduced, N_t depends on the expected token price appreciation μ_t^P. which is also a univariate function of state variable A_t because by Itô's lemma, μ_t^P is equal to $\left(\frac{dP_t/P_t}{dA_t/A_t}\right)\mu^A + \frac{1}{2}\frac{d^2P_t/P_t}{dA_t^2/A_t^2}\left(\sigma^A\right)^2$. In equilibrium, its law of motion is given by a diffusion process

$$d\mu_t^P = \mu_t^{\mu^P}dt + \sigma_t^{\mu^P}dZ_t^A. \tag{46}$$

Now, the dynamics of $\underline{u}_t$ becomes

$$\begin{aligned} d\underline{u}_t = &-\frac{1}{N_t}dN_t + \frac{1}{2N_t^2}\langle dN_t, dN_t\rangle - \frac{1}{A_t}dA_t + \frac{1}{2A_t^2}\langle dA_t, dA_t\rangle \\ &- \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1}{r-\mu_t^P}\right)d\mu_t^P - \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1}{2\left(r-\mu_t^P\right)^2}\right)\left\langle d\mu_t^P, d\mu_t^P\right\rangle \end{aligned} \tag{47}$$

Let $\sigma_t^{\underline{u}}$ denote the diffusion of $\underline{u}_t$. By collecting the coefficients on dZ_t^A in Equation (47), we have

$$\sigma_t^{\underline{u}} = -\frac{\sigma_t^N}{N_t} - \sigma^A - \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\sigma_t^{\mu^P}}{r-\mu_t^P}\right), \tag{48}$$

which, in comparison with Equation (44), contains an extra term that reflects the volatility of expected token price change. Note that, similar to Equation (43), we have

$$dN_t = -g\left(\underline{u}_t\right) d\underline{u}_t - \frac{1}{2} g'\left(\underline{u}_t\right) \left\langle d\underline{u}_t, d\underline{u}_t \right\rangle, \tag{49}$$

so the diffusion of N_t is $-g\left(\underline{u}_t\right) \sigma_t^{\underline{u}}$. Matching it with the conjectured diffusion coefficient σ_t^N gives σ_t^N.

Appendix B - Comparative Statics

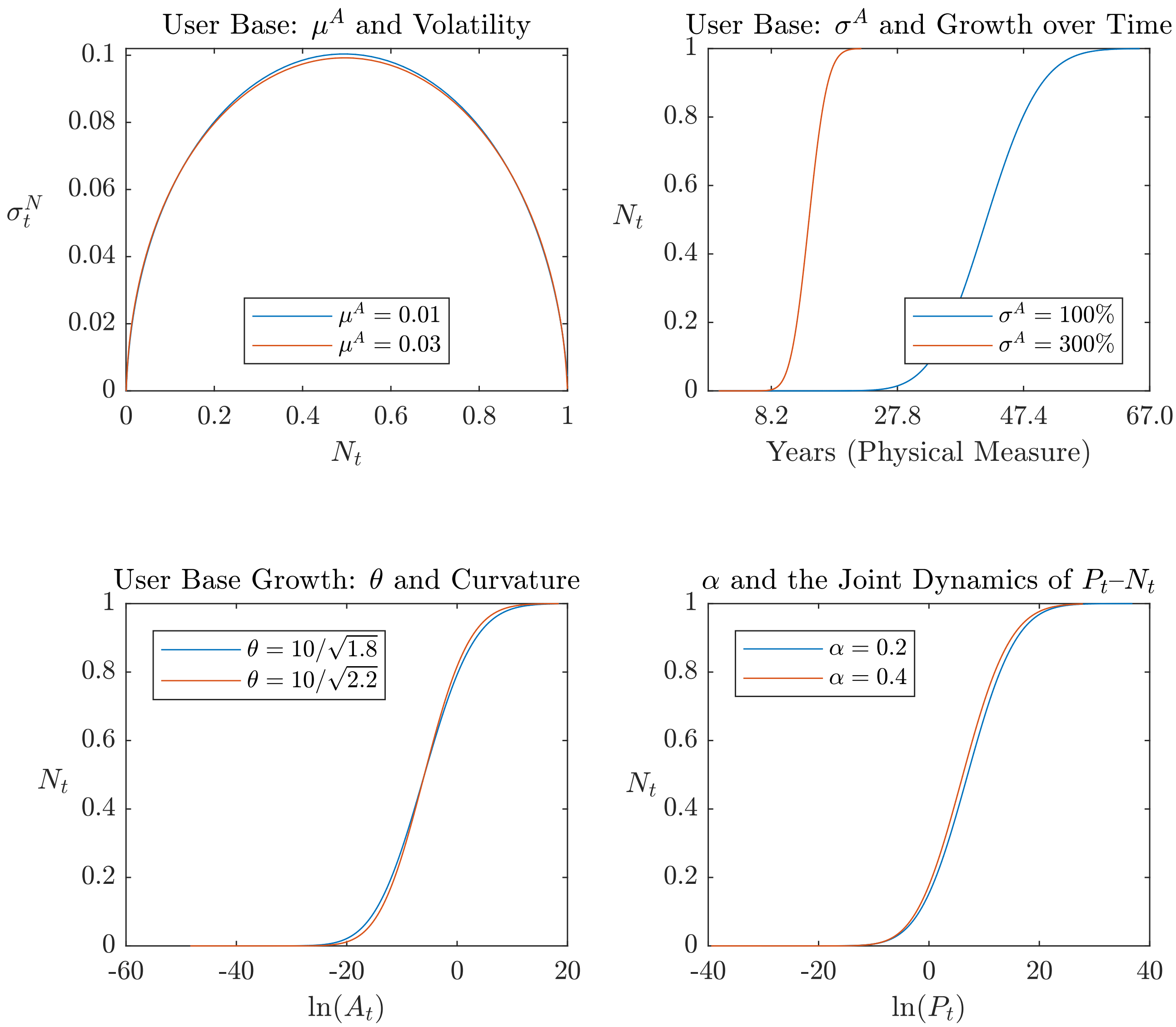

Figure 8: Comparative Statics of Key Parameters.

Appendix C - Model Extensions

C1. Endogenous Growth: from User Base to Productivity

In this paper, our primary focus is on the endogenous and joint dynamics of token price and user base. Through token price that reflects agents expectation, the popularity of a platform in the future increases the present user base. Our analysis thus far has taken the blockchain productivity process as exogenous. In reality, many token and cryptocurrency applications feature an endogenous response of platform productivity to the variation of user base.

A defining feature of blockchain technology is the provision of consensus on decentralized ledgers. In a "proof-of-stake" system, the consensus is more robust when the user base is large and dispersed because no single party is likely to hold a majority of stake; in a "proof-of-work" system, more miners potentially deliver faster and more reliable confirmation of transactions, and miners' participation in turn depends on the size of user base through the associated media coverage (attention in general), transaction fees, and token price. More broadly, A_t represents the general usefulness of the platform. When more users participate, more types of activities can be done on the blockchain. Moreover, a greater user base potentially directs greater resources and research into the blockchain community, accelerating the technological progress.

The endogeneity of blockchain productivity and its dependence on the user base highlight the decentralized nature of this new technology. To reflect this fact and discuss its theoretical implications related to the growth and volatility amplification effects, we modify the process of A_t as follows:

$$\frac{dA_t}{A_t} = \left(\mu_0^A + \mu_1^A N_t\right) dt + \sigma^A dZ_t. \tag{50}$$

By inspection of Equation (1), the definition of trade surplus, it seems that A_t and N_t are not separately identified from the perspective of individual users, because either of the two is simply part of the marginal productivity. However, this argument ignores the fact that by feeding N_t into the process of A_t, the growth rate of A_t is no longer i.i.d.

Consider the case where $d\mu^A(N_t)/dN_t > 0$. A higher level of N_t now induces faster growth of A_t, which leads to a higher level of N_t in the future. Similarly, a lower current level of N_t translates into a downward shifting of the path of N_t going forward. In other words, the endogenous growth of A_t induces *persistence* in N_t. In our benchmark setting, N_t is reset every instant, depending on the exogenous level of A_t. Yet, here path dependence arises, which tends to amplify both the growth and unconditional volatility of N_t by accumulating and propagating shocks to A_t. A formal analysis of this extension is certainly important in the light of improving quantitative performances of the model.

Another way to achieve such path dependence is to assume that agents' decision to join the community or quit incurs an adjustment cost, so N_t becomes the other aggregate state variable, just as in macroeconomic models where capital stock becomes an aggregate state variable when investment is subject to adjustment cost. However, this specification does not capture the endogenous growth of A_t.

C2. Alternative Tokens and Reflecting Boundary

Many blockchain platforms accommodate not only their native tokens but also other cryptocurrencies. For example, any ERC-20 compatible cryptocurrencies are accepted on the Ethereum blockchain.[34] To address this issue, we may consider an alternative upper boundary of A_t. Define ψ as the cost of creating a new cryptocurrency that is perfect substitute with the token we study because it functions on the same blockchain and therefore faces the same common blockchain productivity and agent-specific trade needs. This creates a reflecting boundary at $\overline{A}$ characterized by a value-matching condition and a smooth-pasting condition:

$$P\left(\overline{A}\right) = \psi \text{ and } P'\left(\overline{A}\right) = 0. \tag{51}$$

When token price increases to ψ, entrepreneurs outside of the model will develop a new cryptocurrency that is compatible with the rules of our blockchain system. So, the price level never increases beyond this value. Because it is a reflecting boundary, we need to rule out jumps of token prices, so the first derivative of $P\left(A_t\right)$ must be zero. Again we have exactly three boundary conditions for a second-order ODE and an endogenous upper boundary that uniquely pins down the solution.

Similarly, we may consider potential competing blockchain systems, and interpret ψ as the cost of creating a new blockchain system and its native token, which together constitute

[34] ERC-20 defines a common list of rules for all tokens or cryptocurrencies should follow on the Ethereum blockchain.

a perfect substitute for our current system. This creates the same reflecting boundary for token price. When token price increases to ψ, entrepreneurs outside of the model will build a new system.

Proposition 3 (**Alternative Boundary**). *The upper boundary condition is given by Equation (51) in the two following cases: (1) the blockchain system accepts alternative tokens or cryptocurrencies that can be developed at a unit cost of ψ; (2) an alternative blockchain system that is a perfect substitute of the current system can be developed at a cost of ψ per unit of its native tokens.*

While our framework accommodates the effect of competition, a careful analysis of crypto industrial organization certainly requires more ingredients, especially those that can distinguish between the entry of multiple cryptocurrencies into one blockchain system and competing blockchain systems. Also constituting intersetin future work is the impact of one platform using another platform's native tokens.

C3. Token Supply Schedule

In practice, many cryptocurrencies and tokens feature an increasing supply over time (for example, Bitcoin) or state-contingent supply in order to stabilize token price (for example, Basecoin). Our framework can be modified to accommodate this feature, and thus, serve as a platform for experimenting the impact of token supply on user base growth and token price stability. For example, we may consider the law of motion of token supply M given by an exogenous stochastic process, for example, as follows

$$dM_t = \mu^M (M_t, N_t)\, dt + \sigma^M (M_t, N_t)\, dZ_t^A. \quad (52)$$

A new Markov equilibrium features two aggregate state variable, A_t and M_t.

An alternative formulation entails incrementals of M folloing Poisson-arrivals, as seen in Bitcoin's supply schedule. This formulation has the analytical advantage that equilibria between two Poisson arrivals still have only one state variable A_t. We can solve the model in a backward induction fashion, starting from the asymptotic future where token supply has plateaued and moving back sequentially in the Poisson time given the value function from the previous step.

As in many macroeconomic models, our framework features monetary neutrality: doubling the token supply *from now on* simply reduces token price by half and does not impact any real variables. However, neutrality is only achieved if the change of token supply is implemented uniformly and proportionally for any time going forward. If token supply is adjusted on a contingent basis, agents' expectation of token price appreciation will be affected, through which supply adjustments influence user base, token demand, and the total trade surplus realized on the platform.

Finally, we emphasize that to achieve the desirable effects of a token supply schedule, the schedule must be implemented automatically without centralized third-party interventions, so that dispersed agents take the supply process as given when making decisions. Such commitment to rules and protocols highlights a key difference between cryptocurrency supply and money supply by governments – through the discipline of decentralized consensus, blockchain developers can commit to a token supply schedule.

References

Acemoglu, Daron, Vasco M Carvalho, Asuman Ozdaglar, and Alireza Tahbaz-Salehi, 2012, The network origins of aggregate fluctuations, *Econometrica* 80, 1977–2016.

Athey, Susan, Ivo Parashkevov, Vishnu Sarukkai, and Jing Xia, 2016, Bitcoin pricing, adoption, and usage: Theory and evidence, *Working Paper.*

Bakos, Yannis, and Hanna Halaburda, 2018, The role of cryptographic tokens and icos in fostering platform adoption, .

Balvers, Ronald J, and Bill McDonald, 2017, Designing a global digital currency, *Working Paper.*

Biais, Bruno, Christophe Bisiere, Matthieu Bouvard, and Catherine Casamatta, 2017, The blockchain fold theorem, *Preliminary Work in Progress.*

Canidio, Andrea, 2018, Financial incentives for open source development: the case of blockchain, .

Cao, Sean, Lin William Cong, and Baozhong Yang, 2018, Financial reporting and blockchains: Collaborative auditing, mis-statements, and regulation, *Working Paper.*

Catalini, Christian, and Joshua S Gans, 2018, Initial coin offerings and the value of crypto tokens, Discussion paper, National Bureau of Economic Research.

Chen, Hui, 2010, Macroeconomic conditions and the puzzles of credit spreads and capital structure, *The Journal of Finance* 65, 2171–2212.

Chiu, Jonathan, and Thorsten V Koeppl, 2017, The economics of cryptocurrencies–bitcoin and beyond, *Working Paper.*

Chiu, Jonathan, and Tsz-Nga Wong, 2015, On the essentiality of e-money, Discussion paper, Bank of Canada Staff Working Paper.

Chod, Jiri, and Evgeny Lyandres, 2018, A theory of icos: Diversification, agency, and information asymmetry, .

Ciaian, Pavel, Miroslava Rajcaniova, and dArtis Kancs, 2016, The economics of bitcoin price formation, *Applied Economics* 48, 1799–1815.

Cong, Lin William, and Zhiguo He, 2018, Blockchain disruption and smart contracts, *Forthcoming, Review of Financial Studies.*

——— , and Jiasun Li, 2018, Decentralized mining in centralized pools, *Working Paper.*

Doepke, Matthias, and Martin Schneider, 2017, Money as a unit of account, *Econometrica* 85, 1537–1574.

Easley, David, Maureen O'Hara, and Soumya Basu, 2017, From mining to markets: The evolution of bitcoin transaction fees, *Working Paper.*

Evans, David S, and Richard Schmalensee, 2010, Failure to launch: Critical mass in platform businesses, *Review of Network Economics* 9.

Fernández-Villaverde, Jesús, and Daniel Sanches, 2016, Can currency competition work?, Discussion paper, National Bureau of Economic Research.

Gandal, Neil, and Hanna Halaburda, 2014, Competition in the cryptocurrency market, *Working Paper*.

Gans, Joshua S, and Hanna Halaburda, 2015, Some economics of private digital currency, in *Economic Analysis of the Digital Economy* . pp. 257–276 (University of Chicago Press).

Halaburda, Hanna, and Miklos Sarvary, 2016, Beyond bitcoin, *The Economics of Digital Currencies*.

Harvey, Campbell R, 2016, Cryptofinance, *Working Paper*.

Holden, Craig W, and Avanidhar Subrahmanyam, 2017, A yen for plastic, *Economics Letters* p. 72.

Huberman, Gur, Jacob Leshno, and Ciamac C. Moallemi, 2017, Monopoly without a monopolist: An economic analysis of the bitcoin payment system, working paper 17-92 Columbia Business School.

Jackson, Lloyd K, 1968, Subfunctions and second-order ordinary differential inequalities, *Advances in Mathematics* 2, 307–363.

Li, Jiasun, and William Mann, 2018, Initial coin offering and platform building, *Working Paper*.

Nakamoto, Satoshi, 2008, Bitcoin: A peer-to-peer electronic cash system, *Online Publication*.

Narayanan, Arvind, and Jeremy Clark, 2017, Bitcoin's academic pedigree, *Communications of the ACM* 60, 36–45.

Pagnotta, Emiliano, and Andrea Buraschi, 2018, An equilibrium valuation of bitcoin and decentralized network assets, Working paper Imperial College.

Pástor, Luboš, and Pietro Veronesi, 2009, Technological revolutions and stock prices, *The American Economic Review* 99, 1451–1483.

Rogoff, Kenneth, 2017, Crypto-fools gold?, *Project Syndicate* 9.

Saleh, Fahad, 2017, Blockchain without waste: Proof-of-stake, Discussion paper, working Paper.

Shiller, Robert, 2017, what is bitcoin really worth? don't even ask, *The New York Times* Dec 15.

Sockin, Michael, and Wei Xiong, 2018, A model of cryptocurrencies, *Working Paper*.

Yermack, David, 2017, Corporate governance and blockchains, *Review of Finance (Forthcoming)*.

Yglesias, Matthew, 2012, Social cash: Could facebook credits ever compete with dollars and euros?, *Slate*.