Delegation Uncertainty in the Era of Big Data*

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Abstract

Big data creates a division of knowledge – asset managers use big data and professional techniques to estimate the probability distribution of asset returns, while investors face model uncertainty. Model uncertainty offers a new perspective to understand delegation that, for example, reconciles the growth of asset management industry and its lack of convincing performance. Delegation fundamentally transforms the role of model uncertainty in asset pricing by inducing a hedging motive of investors that increases with the level of delegation. It explains patterns ("anomalies") in the cross-section of asset returns and offers practical guidance to identify alpha that is robust to the rise of arbitrage capital. We provide evidence that supports the assumptions and predictions of our theory.

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1 Introduction

Technological progress is often accompanied by increasing division of labor. The era of big data is defined by exploding data sources and sophisticated data processing techniques, but data analysis requires enormous efforts of professionals. Specialization creates a division of knowledge and induces delegation – we hire professionals to perform tasks using their superior information. However, delegation carries an intrinsic form of uncertainty, even in the absence of moral hazard. The outcome of delegation depends on the professionals' knowledge, which we do not know. This paper studies this delegation uncertainty, and explores its implication on delegated portfolio management and asset pricing.

Asset management industry is being revolutionized by new data sources and statistics toolkits, which help money managers to better estimate the probability distribution of asset returns.[1] In contrast, ordinary investors face the difficulty to gauge probabilities. Our model has two types of agents: managers who know the true return distribution, and investors who face model uncertainty (ambiguity) given by a set of possible probability distributions ("models"). Investors may delegate part of their wealth to managers after paying a fee, while allocate the retained wealth on their own under ambiguity.[2] Given their knowledge of the true return distribution, managers dutifully allocate the delegated wealth on the efficient frontier.[3] The equilibrium expected returns of assets are solved by clearing asset markets, i.e., equating the exogenous supply to the aggregate demand of managers and investors.

Delegation improves investors' welfare by reducing their exposure to the ambiguity in individual assets' returns. As in Gennaioli, Shleifer, and Vishny (2015), this welfare perspective helps understand puzzles in the asset management literature. For example, we characterize the conditions under which delegation happens even though managers underperform the market or deliver zero alpha by holding portfolios proportional to the market.

However, *delegation uncertainty* remains – while managers are committed to deliver the efficient portfolio, the efficient frontier varies across probability models. Therefore, investors hedge the delegation uncertainty when they allocate the retained wealth. Specifically, their portfolio is biased towards (away from) assets whose returns move against (with) the efficient

[1] According to AlternativeData.org, \$400 million were spent on nonstandard data by asset managers in 2017, an increase of 72% from \$232 million in 2016.

[2] The fee may represent a concrete management fee, agency cost, screening cost, or the relative bargaining power of investors over managers that are not explicitly modeled in the paper.

[3] We do not model moral hazard under ambiguity, which is studied by Miao and Rivera (2016).

frontier across models.[4] Delegation hedging generates CAPM alpha in equilibrium, and the cross-section dispersion of alpha depends on investors' model uncertainty and their optimal level of delegation. Intuitively, delegation hedging is stronger when investors choose to delegate more. Therefore, our model identifies a set of assets or trading strategies whose alpha is robust to the rise of arbitrage capital (i.e., wealth allocated by professional managers).

In our model, professional asset managers and investors are different in their knowledge of return distribution. Traditional models typically assume that managers observe a return signal, which is essentially a special case of our setup (i.e., better knowledge of the first moment of distribution). To highlight the division of knowledge, we assume that investors do not learn the probability distribution by observing managers' allocation, and that managers cannot inform investors the distribution as in reality it is difficult for managers to explain the economic rationale or statistical techniques behind investment strategies.[5]

We provide closed-form solutions for investors' delegation and the cross section of expected asset returns by solving a quadratic approximation of investors' utility under ambiguity.[6] As a technical contribution, our approximation extends that of Maccheroni, Marinacci, and Ruffino (2013) into functional spaces. When delegation is unavailable and investors are ambiguity-neutral, our approximation becomes the classic Arrow-Pratt approximation, which generates the mean-variance portfolio of Markowitz (1959) and a CAPM equilibrium.

In out setup, delegation offers investors *model-contingent* allocation of wealth. We abstract away any frictions, so asset managers can be viewed as portfolio formation machines with the knowledge of true return distribution as input and the corresponding efficient portfolio as output.[7] In investors' mind, the overall structure of uncertainty is a two-step lottery: first, a probability model is drawn and the manager observes it and allocates the delegated wealth on the corresponding efficient frontier; second, a state of the world is drawn according to the probability model. Thus, the delegated portfolio is model-contingent, and through it,

[4]This insight is related to Drechsler (2013) who show that investors pay a large premium for index options because they hedge important model misspecification concerns.

[5]Our setup is a special case of model uncertainty in a multi-agent environment studied by Hansen and Sargent (2012) – one type of agents, managers do not face model uncertainty, while the other type do. Learning under model uncertainty (ambiguity) has been studied by Epstein and Schneider (2007), and in the asset pricing literature, Leippold et al. (2008), Ju and Miao (2012) and Choi (2016). Mele and Sangiorgi (2015) study agents' information acquisition under Knightian uncertainty. Ľuboš Pástor and Stambaugh (2012) study how investors' Bayesian learning affect their delegation decision.

[6]We assume smooth ambiguity aversion utility function proposed by Klibanoff, Marinacci, and Mukerji (2005) and Nau (2006) and discussed by Epstein (2010) and Klibanoff, Marinacci, and Mukerji (2012).

[7]The model does not feature moral hazard, heterogeneous manager type, search frictions, and frictions.

the return from delegation is both model- and state-contingent. In contrast, investors' retained wealth is only state-contingent – the return is determined when a state of the world is realized. Delegation uncertainty is due to the model-contingent nature of delegated portfolio. When allocating the retained wealth, investors may hedge such uncertainty by overweighting assets that tend to deliver superior returns under models with inferior frontiers.

The model-contingency induced by delegation has two consequences that help us understand respectively delegation and cross section of asset returns. First, model-contingent allocation improves investors' welfare by allowing them to access efficient portfolio under each probability model. Investors' optimal level of delegation depends on the model uncertainty they face, the cross-model variation of efficient frontier, management fee, and preference parameters, such as risk aversion and ambiguity aversion. We measure investors' model uncertainty by the Bayesian posterior from a latent factor model of stock returns that captures key features of returns from the asset pricing literature. Given the measured uncertainty, the model-implied delegation has 19% correlation with its empirical counterpart.

This new perspective on delegated asset management explains several puzzles in the empirical literature, such as delegation in spite of underperformance relative to indices. First, asset managers can be skilled in knowing higher moments instead of the expected return. Therefore, it is not necessary that ex post, we observe outperformance. Second, investors cannot evaluate fund performances ex ante under rational expectation, so econometricians' performance measurements are based upon an information set different from investors'. How delegation improves welfare depends on the subjective set of candidate probability models that investors entertain. We characterize conditions under which delegation arises even though managers may underperform the market, deliver negative alpha, or simply hold a portfolio proportional to the market portfolio (Fama and French (2010); Lewellen (2011)). Our focus on ex ante welfare echoes that of Gennaioli, Shleifer, and Vishny (2015).

The second consequence of model-contingency through delegation is the investors' hedge against delegation uncertainty. Investors are averse to the cross-model comovement between assets and the efficient frontier (i.e., the delegated portfolio). Hedging against delegation uncertainty induces a two-factor structure in the expected asset returns: a typical CAPM risk premium, and an model uncertainty premium ("alpha"). Alpha is precisely from assets' cross-model comovement with the frontier. We would expect the alpha approaches zero if the economy approaches full delegation (e.g., driven by declining management fees) and managers who do not face model uncertainty dominate asset markets. However, this is not the case. The more investors delegate, the stronger hedge against delegation uncertainty is per dollar of retained wealth. The increasing hedging motive counter-balances the decreasing share of wealth managed by investors under model uncertainty, which sustains the uncertainty premium (alpha). Therefore, our model offers an explanation on why certain investment strategies (e.g., stock-market factors) still deliver alpha in spite of the growing "aribtrage capital" (money managed by professionals).

We test the asset pricing implications of our model in the space of U.S. stock market factors. We focus on factors rather than individual stocks because diversifiable (idiosyncratic) risks should not matter for investors' decisions under any probability distribution. First, we test whether managers have better knowledge of return distribution. If they do, we should observe their portfolio tilt towards factors with superior expected return. Every quarter, we sort factors by their fund ownership (adjusted to match its theoretical counterpart). Factors with high fund ownership consistently outperform those with low fund ownership. Parametric tests based on factor return prediction support this finding of factor timing. A one standard deviation increase of fund ownership adds 1.76% (annualized) to a factor's future return, which translates to a 53% increase over the average factor return in our sample.

Under several assumptions that simplify the structure of investors' model uncertainty, our model predicts that assets' CAPM alpha are proportional to fund managers' ownership. We calculate the CAPM alpha of a portfolio that longs factors with high fund ownership and shorts factors with low fund ownership. The alpha is consistently positive in rolling samples, in spite of the growth of delegation in the past few decades. This is consistent with our prediction that investors' model-hedging motive sustains CAPM alpha even though the wealth managed under ambiguity declines and the delegated share of wealth rises.

Literature. Our paper fits into a broader literature of ambiguity and ambiguity aversion (Hansen and Sargent (2016)).[8] Ambiguity (also called "Knightian uncertainty") is the lack of knowledge of probability distribution and can interpreted as model uncertainty or uncertainty over specific parameters.[9] Ellsberg paradox is one of the most salient examples that

[8] Another related literature studies the "uncertainty shock" and its implications on macroeconomics, for example Bloom (2009) and Jurado et al. (2015) among others.

[9] See Knight (1921) for Knight's well-known distinction between risk (situations in which all relevant events are associated with a unique probability assignment) and uncertainty (situations in which some events do not have an obvious probability assignment).

demonstrate ambiguity-averse behavior. A version of it was noted considerably earlier by John Maynard Keynes in his book "*A Treatise on Probability*" (1921). Widely cited as a fundamental challenge to the expected utility theory, ambiguity aversion has been applied in various fields in economics and finance, especially asset pricing (e.g., Boyarchenko (2012), Easley and O'Hara (2010), Epstein and Wang (1994), Garlappi, Uppal, and Wang (2007), Horvath (2016), Maenhout (2004), Illeditsch (2011), Ilut (2012), Ju and Miao (2012)), contracting (e.g, Fabretti et al. (2014), Miao and Rivera (2016), Rantakari (2008)), real option (Miao and Wang (2011)), corporate governance (Izhakian and Yermack (2017)), and policy intervention during crises (Caballero and Krishnamurthy (2008)). Epstein (2010) and Guidolin and Rinaldi (2010) review the literature.[10]

This paper contributes to the literature of asset pricing theories by offering an alternative decomposition of equilibrium expected return, and show that the price of model uncertainty depends on the endogenous level of delegation. Moreover, we identify a set of assets (or factors) whose CAPM alpha is robust to the growth of professional asset management industry. Guided by the theory, our empirical study contributes to the empirical asset pricing literature. Nagel (2005) show that (unconditional) factor premia in the cross section are most pronounced among stocks with low institutional ownership. We study conditional factor premia, and find that institutional ownership positively forecasts factor returns.

There are many ways to formalize ambiguity and ambiguity aversion.[11] We adopt the smooth ambiguity averse utility function proposed by Klibanoff, Marinacci, and Mukerji (2005) and Nau (2006) because it separates ambiguity from ambiguity aversion (the attitude towards ambiguity).[12] We show that our results hold even when investors are not ambiguity-averse but face ambiguity. In contrast to existing literature on asset pricing under ambiguity, in our setup, ambiguity-neutral investors cannot simply perform Bayesian model-averaging and act as typical risk-averse agents under the average model. This is precisely because through delegation, their return on wealth is both state- and model-contingent, so ambiguity-neutral investors can no longer average out model uncertainty for each state, but instead, are forced to face the joint uncertainty in both state space and model space. Therefore, we

[10] In other areas of economics,

[11] Camerer and Weber (1992), and Wakker (2008) have an explicit focus on defining ambiguity, ambiguity aversion, and how to best model such preferences, with a special focus issues of axiomatization of the resulting criteria and preferences.

[12] Ghirardato, Maccheroni, and Marinacci (2004) take an axiomatic approach to study the separation between ambiguity and agents' attitude towards ambiguity.

are the first to show that delegation arises endogenously from model uncertainty, and at the same time, it fundamentally changes how model uncertainty affects agents' decision making.

Since Jensen (1968), a large literature has documented that active portfolio managers fail to outperform passive benchmarks or to deliver "alpha" to investors.[13] Fama and French (2010) find that the aggregate portfolio of actively managed U.S. equity mutual funds is close to the market portfolio (also Lewellen (2011)), and very few funds produce sufficient benchmark-adjusted returns to cover their costs. Nevertheless, the asset management sector has been growing dramatically. To understand these puzzling findings, this paper proposes an alternative perspective based on welfare improvement (as in Gennaioli, Shleifer, and Vishny (2015)). We characterize the conditions under which managers underperform, deliver negative alpha after fess, and hold portfolio proportional to the market portfolio. Built upon the division of knowledge between professionals and investors on return distribution, our model is complementary to the existing models of delegated asset management (e.g., Berk and Green (2004), Chevalier and Ellison (1999), Guerrieri and Kondor (2012), Ľuboš Pástor and Stambaugh (2012), Kaniel and Kondor (2013), Garleanu and Pedersen (2017), Pástor, Stambaugh, and Taylor (2017) among others).

2 Model

2.1 Model setup

Consider a two-period economy where agents make decisions in the first period, and asset returns are realized in the second and final period. There are N risky assets, whose returns are stacked in a vector $\mathbf{r} = \{r_i\}_{i=1}^N$, and one risk-free asset that delivers a risk-free return r_f. Define $\mathbf{\Omega}$ as the set of states of the world in the final period, so the vector of asset returns is a mapping from the state space to real numbers, $\mathbf{r} : \mathbf{\Omega} \mapsto \mathbf{R}^N$.

There are a unit mass of homogeneous investors, and a unit mass of homogeneous fund managers. For simplicity, we assume that each investor is matched with one fund manager. Later, we discuss how our results can be extended to more general settings.

Model uncertainty and preference. A representative investor is endowed with one unit

[13]See Barras, Scaillet, and Wermers (2010), Carhart (1997), Del Guercio and Reuter (2014), Fama and French (2010), Gruber (1996), Malkiel (1995), Wermers (2000), among others.

of wealth. She chooses δ, which is the fraction of wealth invested in the fund. We specify the delegation return later after laying out the investor's information set and preference. The investor needs to allocate $1-\delta$ retained wealth, and chooses $\mathbf{w}^o$ (superscript "o" for "own" allocation), which is a column vector of portfolio weights on the N risky assets. The investor does not know the return distribution, so she forms her own portfolio under model uncertainty (or ambiguity). Here ambiguity and model uncertainty are used interchangeably.

Model uncertainty is given by Δ, a non-singleton set of candidate probability distributions of $\mathbf{r}$ ("models"). For a probability measure $Q \in \Delta$, the investor assigns a prior $\pi(Q)$, which is the *subjective* probability that Q is the true return distribution.

The investor's preference is represented by the smooth ambiguity-averse utility function in Klibanoff, Marinacci, and Mukerji (2005) ("KMM"). The purpose of using this specification is to separate ambiguity itself and the aversion to ambiguity.[14] Utility is defined over the terminal wealth, $r_{\delta,\mathbf{w}^o,\mathbf{w}^d}$, whose subscripts show the dependence on the delegation level δ, the investor's own portfolio $\mathbf{w}^o$, and the delegation portfolio chosen by the manager $\mathbf{w}^d$ (superscript "d" for "delegation") that we introduce shortly. The utility function is

$$V\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right) = \int_\Delta \phi\left(\int_\Omega u\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right) dQ(\omega)\right) d\pi(Q) \tag{1}$$

$\phi(\cdot)$ and $u(\cdot)$ are strictly increasing functions and twice continuously differentiable. Concavity of $u(\cdot)$ and $\phi(\cdot)$ represent risk and ambiguity aversion respectively.

Delegation as model-contingent allocation. Fund managers' preference is not modeled. A representative manager does not make any decision other than constructing an efficient portfolio under his knowledge of P, the true probability distribution of $\mathbf{r}$. We may think of a fund manager as a portfolio formation machine that creates a vector of portfolio weights $\mathbf{w}^d$ that achieves the efficient frontier (more details later on the definition of efficient portfolio).

To access this "machine", the investor pays an exogenous proportional fee ψ. In a richer setting, ψ can be determined by the competition between fund managers, a manager's effort cost (and asset management technology), agency cost, and bargaining power.

What can a fund manager offer? From the investor's perspective, for any candidate

[14] Epstein (2010) has drawn the attention to the fact that KMM framework may imply counterintuitive behaviors, but Klibanoff, Marinacci, and Mukerji (2012) have replied that those Ellsberg-style thought experiments do not pose difficulty for the smooth ambiguity model.

model $Q \in \Delta$, if it is the true model, the manager knows it and constructs the *corresponding* efficient portfolio $\mathbf{w}^d(Q)$. Therefore, delegation makes investors' wealth *model-contingent.* This is shown clearly once we write out the total return on the investor's wealth,

$$\begin{aligned} r_{\delta,\mathbf{w}^o,\mathbf{w}^d} &= (1-\delta)\left[r_f + (\mathbf{r} - r_f\mathbf{1})^T \mathbf{w}^o\right] + \delta\left[r_f + (\mathbf{r} - r_f\mathbf{1})^T \mathbf{w}^d(Q)\right] \\ &= r_f + (\mathbf{r} - r_f\mathbf{1})^T \left[(1-\delta)\mathbf{w}^o + \delta\mathbf{w}^d(Q)\right],\ Q \in \Delta. \end{aligned} \tag{2}$$

The investor's own portfolio is a N-dimensional *vector*, $\mathbf{w}^o \in \mathbf{R}^N$. In contrast, the delegated portfolio, $\mathbf{w}^d$, is a *mapping* from the model space to real numbers, $\mathbf{r} : \Delta \mapsto \mathbf{R}^N$, because if any Q is the true model, the manager constructs the corresponding efficient portfolio $\mathbf{w}^d(Q)$. Through delegation, the total return is a mapping from the state space *and* the model space to real numbers, $r_{\delta,\mathbf{w}^o,\mathbf{w}^d} : \Omega \times \Delta \mapsto \mathbf{R}$. If $\delta = 0$, the portfolio return is $r_f + (\mathbf{r} - r_f\mathbf{1})^T \mathbf{w}^o$, which just a mapping from the state space $\mathbf{\Omega}$ to $\mathbf{R}$.

Delegation improves welfare through model-contingent allocation. As in Segal (1990), let us consider an imaginary economy with two stages: (1) investors choose $\mathbf{w}^o$ and δ but cannot bet on which probability model is true (the first-stage "state"); (2) the model is drawn and known by managers who allocate the delegated wealth. Here, model uncertainty translates into a form of market incompleteness that can be reduced by delegation.[15] Later we show that this welfare benefit is key to reconcile the sizable delegation and mediocre fund performances in data.

Delegation fundamentally changes the nature of ambiguity and how it enters into investors' portfolio choice. The delegated portfolio, $\mathbf{w}^d(Q)$, varies across probability models. This "delegation uncertainty" gives rise to a hedging motive – the cross-model comovement between $\mathbf{w}^d(Q)$ and an asset's return distribution becomes a key consideration in investors' portfolio decision. Without delegation, the return on investors' wealth does not vary with the probability model and this hedging motive disappears. In Section 2.4, we show that investors' cross-model hedging motive in $\mathbf{w}^o$, induced by delegation, generates a two-factor structure of asset returns in equilibrium. This motive becomes stronger when the delegation level is higher, so the equilibrium never converges to CAPM (a single-factor structure) even if δ approaches 100% and only managers trade assets.

We will show that this hedging motive even appears in the portfolio choice of *ambiguity-*

[15]This discussion is in line with Maenhout (2004) and Strzalecki (2013) who show an intrinsic link between ambiguity aversion and the preference for early resolution of risk (e.g., Epstein and Zin (1989)).

neutral investors (with linear $\phi(\cdot)$), so the two-factor structure of asset market equilibrium does not require ambiguity aversion, which stands in contrast with existing asset pricing models with ambiguity (e.g., Kogan and Wang (2003)). In other words, once model uncertainty manifests into delegation uncertainty, it matters for asset pricing even without ambiguity aversion. Note that without delegation, ambiguity-neutral investors simply perform model-averaging because the return on wealth is only state-dependent, instead of state- and model-dependent. They calculate π-weighted average of probabilities of any event,

$$\overline{Q}(A) = \int_{Q \in \Delta} Q(A)\, d\pi(Q), \text{ for any } A \subset \Omega. \tag{3}$$

Under this "average model", ambiguity-neutral investors form a portfolio, behaving as typical expected-utility agents, and do not hedge model uncertainty when delegation is unavailable.

Before the formal analysis, several observations are in order. First, very importantly in our setting, managers do not directly inform their investors which model is true. Otherwise, the delegation uncertainty disappears. This reflects the realistic difficulty of communication between professional managers and investors. Particularly, big data and sophisticated techniques equip fund managers with increasingly advanced tools to understand return distribution, but at the same time, create a division of knowledge. It is increasingly difficult for investors to understand the information set and techniques of professional asset managers.

Our setup nests typical models in the literature of delegated portfolio management as special cases, where managers obtain predictive signals, i.e., better knowledge of the first moment of return distribution. Here we study the most general form of skills – distribution knowledge. Busse (1999) finds volatility-timing ability of mutual fund managers (Chen and Liang (2007) for hedge funds).[16] Jondeau and Rockinger (2012) study the economic value added by forecasting up to the fourth moments of returns ("distribution timing"). As the asset management industry increasingly leverages on big data and nonlinear data processing techniques, such as machine learning, it is important to model asset management under this generic specification of skills. As will be shown later, the model sheds light on many issues on delegated portfolio management and asset pricing.

[16] In line with the evidence, Ferson and Mo (2016) provide a framework to evaluate portfolio performance in both market timing and volatility timing.

2.2 A quadratic approximation

To solve the investor's delegation and portfolio allocation in closed forms, we approximate the utility function in a quadratic fashion by extending the results of Maccheroni, Marinacci, and Ruffino (2013) ("MMR") into functional spaces. MMR does not allow agents' wealth to be model-contingent. Model-contingent allocation through delegation is the key in our model. In this paper, we adopt their technical regularity conditions and the approximation conditions. We will show that our approximation nests MMR's as a special case.

First, we define the certainty equivalent.

Definition 1 *A representative investor's certainty equivalent is defined by*

$$C\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right) = \upsilon^{-1}\left(\int_{\Delta} \phi\left(\int_{\Omega} u\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right) dQ\left(\omega\right)\right) d\pi\left(Q\right)\right), \tag{4}$$

where υ is a composite function $\upsilon = \phi \circ u$.

Accordingly, we write the investor's delegation and portfolio problem as follows:

$$\max_{\mathbf{w}^o,\delta} \left\{C\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right) - \psi\delta\right\} \tag{5}$$

where the return on wealth, $r_{\delta,\mathbf{w}^o,\mathbf{w}^d}$, is both state- and model-contingent (Equation (2)), and investors pay a proportional asset management fee ψ.

The quadratic form is similar to the mean-variance preference but incorporates both risk and ambiguity. We define two parameters of risk aversion and ambiguity aversion respectively in a small neighborhood of the return on wealth around risk-free rate r_f.

Definition 2 *At risk free return r_f, the local absolute risk aversion γ is defined as*

$$\gamma = -\frac{u''\left(r_f\right)}{u'\left(r_f\right)} \tag{6}$$

and marginal-utility-adjusted local ambiguity aversion θ is defined as

$$\theta = -u'\left(r_f\right)\frac{\phi''\left(u\left(r_f\right)\right)}{\phi'\left(u\left(r_f\right)\right)} \tag{7}$$

Before the quadratic representation of investors' preference, we introduce notations:

- Define q as the Radon-Nikodym derivative of Q w.r.t. $\overline{Q}$, i.e., $q(\omega) = \frac{dQ(\omega)}{d\overline{Q}(\omega)}$ for $\omega \in \Omega$. q and Q are used interchangeably to represent a candidate probability model in Δ.

- Let $R^{\mathbf{w}} = (\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}$ denote the excess return of any portfolio $\mathbf{w}$.

- Let $R_Q^{\mathbf{w}} = E_Q\left[(\mathbf{r} - r_f \mathbf{1})^T \mathbf{w}\right]$ denote the expectation of excess return of $\mathbf{w}$ under Q.

- Given $Q \in \Delta$, let $E_Q(X)$ and $\sigma_Q^2(X)$ denote the expectation and variance of any random variable X respectively, and μ_Q^X and Σ_Q^X denote the vector of expectation and the matrix of covariance of any random vector respectively.

- Given $Q \in \Delta$, the covariance of two random variables X and Y is denoted by $cov_Q(X, Y)$.

Quadratic Preference. Using the Taylor expansion in the functional space, we approximate the certainty equivalent as in Proposition 1. The proof uses the generalized Fréchet derivatives in the Banach spaces. Details are provided in the Appendix.

Proposition 1 (Quadratic preference) *The smooth ambiguity-averse preference over the state- and model-contingent return, $r_{\delta,\mathbf{w}^o,\mathbf{w}^d}$, i.e. mappings from $\mathbf{\Omega} \times \mathbf{\Delta}$ to $\mathbf{R}$, can be represented by the certainty equivalent, which has the following expansion:*

$$
\begin{aligned}
C\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right) = & r_f + (1-\delta)^2 R_Q^{\mathbf{w}^o} - \frac{(1-\delta)^2}{2}\left(\gamma\sigma_Q^2\left(R^{\mathbf{w}^o}\right) + \theta\sigma_\pi^2\left(R_Q^{\mathbf{w}^o}\right)\right) + \\
& \delta E_\pi\left(R_Q^{\mathbf{w}^d}\right) - \frac{\delta^2}{2}\left[\gamma E_\pi\left(\sigma_Q^2\left(R^{\mathbf{w}^d}\right)\right) + \theta\sigma_\pi^2\left(R_Q^{\mathbf{w}^d}\right)\right] \\
& - (\theta+\gamma)(1-\delta)\delta cov_\pi\left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d}\right) + \boldsymbol{R}\left(\mathbf{w}^o, \mathbf{w}^d\right),
\end{aligned}
\tag{8}
$$

where $R\left(\mathbf{w}^o, \mathbf{w}^d\right)$ is a high-order term that satisfies $\lim_{\left(\mathbf{w}^o,\mathbf{w}^d\right)\to\mathbf{0}} \frac{\boldsymbol{R}\left(\mathbf{w}^o,\mathbf{w}^d\right)}{\left\|\left(\mathbf{w}^o,\mathbf{w}^d\right)\right\|^2} = 0.$

Following MMR, we use the same approximation condition – if portfolio is sufficiently diversified such that its matrix norm is close to zero, the residual term can be ignored.[17] In the following, we use this second-order approximation in investors' objective function. The local quadratic approximation allows us to intuitively understand the investor's preference.

[17]Our convergence criterion differs from that of Garlappi and Skoulakis (2011), who study how the accuracy of polynominal approximations to expected utility depends on the included Taylor expansion terms. Hlawitschka (1994) show (i) when a Taylor series of expected utility diverges, then truncated Taylor series—particularly second-order expansions—may provide excellent approximations for the purpose of portfolio selection; (ii) when a Taylor series does converge, adding more terms may worsen the approximation.

As previously defined, $R_{\overline{Q}}^{\mathbf{w}^o}$ is the expected excess return to her own portfolio $\mathbf{w}^o$ under the average model $\overline{Q}$. An increases in $R_{\overline{Q}}^{\mathbf{w}^o}$ leads to higher utility, but the sensitivity, $(1-\delta)^2$, decreases in the level of delegation δ. $\sigma_{\overline{Q}}^2\left(R^{\mathbf{w}^o}\right)$ is the variance of excess return to the own portfolio under the average model $\overline{Q}$. As a measure of risk, it decreases utility. The sensitivity to risk increases in γ, the parameter of risk aversion. $\sigma_\pi^2\left(R_Q^{\mathbf{w}^o}\right)$ measures model uncertainty. It is the *cross-model* variation of the *expected excess return*, as $R_Q^{\mathbf{w}^o}$ denotes the expected return on the investor's retained wealth under a particular model Q. The sensitivity to ambiguity increases in θ, the parameter of ambiguity aversion. As δ increases, and thus, the retained wealth decreases, both sensitivities to risk and ambiguity decline.

The delegation return enters into the utility in an intuitive manner. $E_\pi\left(R_Q^{\mathbf{w}^d}\right)$ is the expected excess return of the delegated portfolio, averaged over models under prior π,

$$E_\pi\left(R_Q^{\mathbf{w}^d}\right) = \int_{Q\in\Delta} E_Q\left[(\mathbf{r}-r_f\mathbf{1})^T\,\mathbf{w}^d(Q)\right] d\pi(Q),$$

where $R_Q^{\mathbf{w}^d}$ is the expected excess return of delegated portfolio if Q is the true model. Utility increases in the cross-model average of expected return to delegation. $\sigma_\pi^2\left(R_Q^{\mathbf{w}^d}\right)$ measures the *ambiguity* in delegation return. It is a cross-model variance of *expected* excess return from delegation, so it reduces utility, and its sensitivity increases in the level of delegation δ and ambiguity aversion θ. $E_\pi\left(\sigma_Q^2\left(R^{\mathbf{w}^d}\right)\right)$ measures the *risk* in delegation return averaged over models, as $\sigma_Q^2\left(R^{\mathbf{w}^d}\right)$ is the variance of delegation return under a particular Q. Intuitively, the sensitivity to delegation risk increases in risk aversion γ.

The terms discussed so far can be summarized into two categories. First, averaging over models, what are the expected returns and return variances ("risk"). Second, the cross-model mean and variance of the *expected* returns under prior π over the model space Δ ("ambiguity"). The quadratic approximation shows how the these statistics enter into utility, and how the utility sensitivities to these statistics depend on risk aversion, ambiguity aversion, and the level of delegation.

The last term in the quadratic form deserves more attention. It is the cross-model covariance between the expected delegation return and the expected return on retained wealth. Investors do not treat the delegation return and their own investment opportunity set separately, but instead, they want to hedge the cross-model uncertainty. Specifically, if an asset tends to deliver a higher expected return under models where the expected delegation return is low, then investors would like to invest more in this assets. As long as $\delta < 100\%$, the investor has to deal with the cross-model uncertainty from delegation when allocating retained wealth. $cov_\pi \left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d} \right)$ precisely captures such cross-model *hedging motive*.

This hedging term has a utility sensitivity that increases in both risk aversion γ and ambiguity aversion θ. Given γ and θ, the sensitivity is maximized at $\delta = \frac{1}{2}$. Intuitively, the investor cares the most about the comovement between the delegation performance and the return on her retained wealth, when she divides wealth 50/50. As will be shown later, this hedging motive has critical implications on the equilibrium expected returns of risky assets.

Our quadratic approximation nests MMR's solution (when $\delta = 0$, i.e., no delegation) and the standard mean-variance preference (when $\delta = 0$ and $\theta = 0$, i.e., no delegation and no ambiguity aversion) as special cases.

Corollary 1 *Without delegation, i.e., $\delta = 0$, the approximation degenerates to the quadratic approximation of smooth ambiguity utility by Maccheroni, Marinacci, and Ruffino (2013):*

$$C \left(r_f + (\mathbf{r} - r_f \mathbf{1})^T \left[(1-\delta) \mathbf{w}^o + \delta \mathbf{w}^d (Q) \right] \right) \approx r_f + R_{\overline{Q}}^{\mathbf{w}^o} - \frac{\gamma}{2} \sigma_{\overline{Q}}^2 \left(R^{\mathbf{w}^o} \right) - \frac{\theta}{2} \sigma_\pi^2 \left(R_Q^{\mathbf{w}^o} \right). \quad (9)$$

If $\delta = 0$ and $\theta = 0$, the quadratic form degenerates to the standard mean-variance utility under the average model $\overline{Q}$:

$$r_f + R_{\overline{Q}}^{\mathbf{w}^o} - \frac{\gamma}{2} \sigma_{\overline{Q}}^2 \left(R^{\mathbf{w}^o} \right). \quad (10)$$

Later, we show that the investor's optimal portfolio choice $\mathbf{w}^o$ nests MMR's solution of optimal portfolio and the mean-variance portfolio of Markowitz (1959) as special cases.

Delegation portfolio. To derive the solution to the investor's problem and equilibrium asset pricing implications, we need to specify the delegation portfolio. In line with Corollary 1, the investor informs her risk aversion to the fund manager, and the manager forms the mean-variance efficient portfolio given his knowledge of the true distribution of $\mathbf{r}$. Therefore, in the investor's mind, for any $Q \in \Delta$, the managers solves

$$\max_{\mathbf{w}^d} \left\{ \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1} \right)^T \mathbf{w}^d - \frac{\gamma}{2} \left(\mathbf{w}^d \right)^T \Sigma_Q^{\mathbf{r}} \left(\mathbf{w}^d \right) \right\}$$

where, as previously defined, $\mu_Q^{\mathbf{r}}$ and $\Sigma_Q^{\mathbf{r}}$ are the mean vector and covariance matrix of $\mathbf{r}$

under probability measure Q. The delegated portfolio is *model-contingent*, $\mathbf{w}^d : \Delta \mapsto \mathbf{R}^N$:

$$\mathbf{w}^d(Q) = \left(\gamma \Sigma_Q^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1}\right). \tag{11}$$

Under Gaussian asset returns and CARA $u(\cdot)$ with absolute risk aversion γ, $\mathbf{w}^d(Q)$ is the exact maximizer of $u(\cdot)$ for any given Q. Even without ambiguity aversion (i.e., under linear $\phi(\cdot)$), as long as $\phi'(\cdot) > 0$, the investor always achieves higher utility by delegating asset allocation to a fund manager who efficiently allocates wealth for *each* candidate model.

2.3 Investor optimization

Investor portfolio choice. We solve the optimal level of delegation δ and portfolio $\mathbf{w}^o$ by maximizing the quadratic approximation given by Equation (8). Proposition 2 gives the investor's choice of own portfolio of risky assets, $\mathbf{w}^o$. Details are provided in the Appendix.

Proposition 2 (Investor portfolio under ambiguity & delegation) *Given the optimal level of delegation δ, the investor's own portfolio of risky assets is given by*

$$\mathbf{w}_\delta^o = \left(\gamma \Sigma_{\overline{Q}}^{\mathbf{r}} + \theta \Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)^{-1}\left[\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right) - \underbrace{(\theta+\gamma)\left(\frac{\delta}{1-\delta}\right) cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)}_{\textit{uncertainty hedging demand}}\right]. \tag{12}$$

If the investor could not delegate ($\delta = 0$), her portfolio would be

$$\mathbf{w}_0^o = \left(\gamma \Sigma_{\overline{Q}}^{\mathbf{r}} + \theta \Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right),$$

where the subscript "0" represent "zero" delegation. This is also MMR's solution of ambiguity investor's portfolio problem.[18] $\Sigma_{\overline{Q}}^{\mathbf{r}}$ measures risk, the covariance matrix of asset returns under the average model $\overline{Q}$. It enters into the optimal portfolio scaled by γ, the parameter of risk aversion. In contrast, $\Sigma_\pi^{\mu_Q^{\mathbf{r}}}$ is the cross-model covariance matrix of *expected* asset return vector $\mu_Q^{\mathbf{r}}$. It measures ambiguity. The optimal portfolio's sensitivity to $\Sigma_\pi^{\mu_Q^{\mathbf{r}}}$ depends on θ, the parameter of ambiguity aversion. If $\theta = 0$, the optimal portfolio becomes the standard

[18] Garlappi, Uppal, and Wang (2007) derive a similar portfolio by incorporating estimation errors in expected returns (a maxmin approach in the spirit of Gilboa and Schmeidler (1989)).

formula by Markowitz (1959) under the average model, i.e. $\left(\gamma\Sigma_{\overline{Q}}^{\mathbf{r}}\right)^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)$. Without delegation, ambiguity-neutral investors use Bayesian model averaging.

Given $\delta > 0$, the portfolio exhibits a hedging demand from $cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)$, the cross-model comovement between the expected excess returns of assets, $\mu_Q^{\mathbf{r}}$, and the expected excess return from delegation, $R_Q^{\mathbf{w}^d}$. The investor knows that whichever model is true, the fund manager must know it and construct the efficient portfolio accordingly, but the true model is still unknown. Therefore, the investor must design her own portfolio in a way that is "robust" to such ambiguity. The higher the ambiguity aversion is, the more sensitive the investor's portfolio choice to this covariance term.

Even if we shut down ambiguity aversion ($\theta = 0$), we still have the hedging demand, which is $-\gamma\left(\frac{\delta}{1-\delta}\right) cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)$, depending on the risk aversion parameter. Fund managers select the mean-variance efficient portfolio for investors for each model, but the investors still have allocate the retained wealth. To do that, they must consider all the probability models and make their own portfolio robust to the cross-model variation in investment opportunity set and delegated return. This cross-model hedging motive moves the investor's total portfolio away from the efficient frontier *within* each particular model, so higher *risk* aversion makes investors more cautious to the cross-model covariance between asset returns and delegation return.

Let $cov_\pi\left(\mu_Q^{\mathbf{r}_i}, R_Q^{\mathbf{w}^d}\right)$ denote the i-th element of $cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)$. It represents the covariance between asset i's expected return and the delegation return. When the expected delegation return comoves with asset i's expected return, i.e. $cov_\pi\left(\mu_Q^{\mathbf{r}_i}, R_Q^{\mathbf{w}^d}\right) > 0$, the investor reduces investment in asset i. When asset i's expected return moves against the expected delegation return, i.e. $cov_\pi\left(\mu_Q^{\mathbf{r}_i}, R_Q^{\mathbf{w}^d}\right) < 0$, the investor demands more of asset i as if buying an insurance against delegation uncertainty. This hedging motive will have critical implications on the equilibrium cross-section of expected asset returns.

Optimal delegation. The optimal fraction of wealth delegated to fund managers depends on the structure of investors' ambiguity and delegation fee ψ.

Proposition 3 (Optimal delegation given $\mathbf{w}^o$) *Given the optimal portfolio* $\mathbf{w}^o$*, the in-*

vestor's optimal delegation level δ is given by the first order condition:

$$\delta = \frac{E_\pi\left(R_Q^{\mathbf{w}^d}\right) - R_{\overline{Q}}^{\mathbf{w}^o} - (\theta+\gamma)\, cov_\pi\left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d}\right) - \psi}{E_\pi\left(R_Q^{\mathbf{w}^d}\right) - R_{\overline{Q}}^{\mathbf{w}^o} - (\theta+\gamma)\, cov_\pi\left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d}\right) + \theta\sigma_\pi^2\left(R_Q^{\mathbf{w}^d}\right)}. \tag{13}$$

The solution is very intuitive. If the investor can achieve a high return on her own, (i.e. high $R_{\overline{Q}}^{\mathbf{w}^o}$), delegation decreases. If the expected return on retained wealth $R_Q^{\mathbf{w}^o}$ comoves closely with the expected return on delegated wealth $R_Q^{\mathbf{w}^d}$ across models (i.e. high $cov_\pi\left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d}\right)$), delegation also decreases. The investor are averse to the cross-model comovement, as reflected in the choice of $\mathbf{w}^o$. Delegation will increase if the delegation return is expected to be high across models (i.e. high $E_\pi\left(R_Q^{\mathbf{w}^d}\right)$), and if it does not fluctuate much across probability models (i.e. low $\sigma_\pi^2\left(R_Q^{\mathbf{w}^d}\right)$). Note that the investor's own portfolio $\mathbf{w}^o$ depends on δ, so Equation (13) only implicitly defines δ. The next corollary solves δ explicitly as a function of the investor's ambiguity structure and management fee.

Corollary 2 (Optimal delegation) *The investor's optimal delegation level δ is given by*

$$\delta = \frac{E_\pi\left(R_Q^{\mathbf{w}^d}\right) - (\theta+\gamma)\, B - C - \psi}{E_\pi\left(R_Q^{\mathbf{w}^d}\right) + \theta\sigma_\pi^2\left(R_Q^{\mathbf{w}^d}\right) - (\theta+\gamma)^2 A - 2(\theta+\gamma)\, B - C}, \tag{14}$$

where

$$A = cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)^T \left(\gamma\Sigma_{\overline{Q}}^{\mathbf{r}} + \theta\Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)^{-1} cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right), \tag{15}$$

$$B = cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)^T \left(\gamma\Sigma_{\overline{Q}}^{\mathbf{r}} + \theta\Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)^{-1} \left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right), \tag{16}$$

$$C = \left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)^T \left(\gamma\Sigma_{\overline{Q}}^{\mathbf{r}} + \theta\Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)^{-1} \left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right). \tag{17}$$

The solution in Equation (14) depends on the complicated structure of the investor's model uncertainty that involves the cross-model mean and variance of expected delegation return and the cross-model comovement of delegation return and asset returns.[19] In

[19] To solve δ, we substitute the investor's optimal portfolio into Equation (13), so the formula is solved under the assumption of an interior solution, i.e., $\delta < 1$. When $\delta = 1$ and the investor does not retain any wealth to manage on her own, the investor's optimal portfolio given by Equation (12) is not well defined. This explains why even if delegation is free (i.e., $\psi = 0$), Equation (14) does not give 100% delegation. Intuitively, since the manager forms the efficient portfolio under each probability model, the investor with quadratic utility should fully delegate when $\psi = 0$. Therefore, the complete solution of delegation should be 100% if $\psi = 0$, and the interior value given by Equation (14) if $\psi > 0$.

Section 3.3, we estimate a representative investor's model uncertainty and calculate the model-implied delegation using this solution. We show that the model-implied δ has a 19% correlation with the data counterpart.

Comparative statics under simplified model uncertainty. We derive comparative statics and explore more economic intuitions under a simplified structure of model uncertainty. We make the following assumptions that correspond to typical settings where delegated portfolio management has been studied – professional asset managers obtain return signals, i.e., superior knowledge on the first moment of asset returns. Accordingly, investors face uncertainty in their expectation of future asset returns.

Assumption 1 *The investor knows the true covariance matrix: for any* $Q \in \Delta$, $\Sigma_Q^{\mathbf{r}} = \Sigma_P^{\mathbf{r}}$.

Under this assumption and the quadratic approximation of investor preference, the model uncertainty is only about the expected returns, which is captured by the subjective covariance matrix of expected returns, $\Sigma_\pi^{\mu_Q^{\mathbf{r}}}$, given prior π over candidate models. If the investor's model uncertainty is from estimation errors, the diagonal of $\Sigma_\pi^{\mu_Q^{\mathbf{r}}}$ records the squared standard errors of the expected return estimator, which naturally depends on the volatility (and covariance) of returns under the true model (i.e., data generating process). Therefore, we add the following assumption on π that links model uncertainty to volatility.

Assumption 2 *The investor's subjective belief of expected return is given by a normal distribution, whose covariance is proportional to the true return variance:*

$$\mu_Q^{\mathbf{r}} \sim N\left(\mu_{\overline{Q}}^{\mathbf{r}}, \upsilon \Sigma_P^{\mathbf{r}}\right). \tag{18}$$

Since $\mu_Q^{\mathbf{r}} \sim N\left(\mu_{\overline{Q}}^{\mathbf{r}}, \upsilon \Sigma_P^{\mathbf{r}}\right)$, υ that parameterizes the level of model uncertainty. This setup can be easily understood as "parameter uncertainty" or "estimation error" when the investor tries to estimate the expected excess returns, which coincides with the interpretation of ambiguity by Bewley (2011).[20] The normality assumption of the prior over $\mu_Q^{\mathbf{r}}$ also brings technical convenience. As shown in Appendix C, we can apply the Isserlis' theorem to dramatically simplify investors' optimal delegation and portfolio choice.

[20] Boyle et al. (2012), and Ilut and Schneider (2014) also introduce ambiguity through uncertainty in the mean.

$N\left(\mu_{\overline{Q}}^{\mathbf{r}}, \upsilon\Sigma_P^{\mathbf{r}}\right)$ is the popular *conjugate prior*. υ can be understood as the inverse of the size of estimation sample. If the investor has T observations of $\mathbf{r}$ and she assumes the independence across observations, the method-of-moment estimator of the expected return is $\frac{1}{T}\Sigma_{t=1}^{T}\mathbf{r}$ and its covariance is $\frac{1}{T}\Sigma_P^{\mathbf{r}}$. This case directly applies to $\Sigma_{\pi}^{\mu_Q^{\mathbf{r}}} = \upsilon\Sigma_P^{\mathbf{r}}$ with $\upsilon = \frac{1}{T}$. Larger υ means smaller sample and larger estimation error (or ambiguity). It is natural to assume that $\upsilon < 1$ under this interpretation, because $\frac{1}{T} < 1$ for non-singleton sample.

Assumption 3 *$\upsilon < 1$.*

These assumptions highlight the link between volatility and ambiguity. When assuming the covariance of asset returns are known to investors, larger volatility means the expected returns are harder to estimate (higher parameter uncertainty). This model suggests that delegation should also relate to the potentially time-varying uncertainty induced by the evolution of asset return volatility. The case of known covariance and unknown expected returns echoes the observation by Merton (1980). Kogan and Wang (2003) also consider this case in their discussion of portfolio selection under ambiguity.

Using these assumptions, we solve explicitly the optimal delegation as a function of the exogenous parameters, and simplifies the formula of optimal portfolio choice. We provide derivation details in the Appendix, but the main intuition can be simply understood by noticing that for any model Q, the expected delegation return can be decomposed as follows,

$$
\begin{aligned}
R_Q^{\mathbf{w}^d} &= \left(\mu_Q^{\mathbf{r}} - r_f\mathbf{1}\right)^T \mathbf{w}^d(Q) = \left(\mu_Q^{\mathbf{r}} - r_f\mathbf{1}\right)^T \left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}} - r_f\mathbf{1}\right) \\
&= \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)^T \left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right) + \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)^T \left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right) + \\
&\quad \left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)^T \left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right) + \left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)^T \left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right), \\
&= \underbrace{\left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)^T \left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)}_{\textit{Chi-squared distribution}} + \underbrace{2\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)^T \left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}}_{\textit{constant vector}} \underbrace{\left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)}_{\textit{Normal distribution}} + \underbrace{R_{\overline{Q}}^{\mathbf{w}^d}}_{\textit{constant}},
\end{aligned}
$$

where the distributional properties labeled below each term are obtained under the assumption that investors' prior π is Gaussian, i.e., $\mu_Q^{\mathbf{r}} \sim N\left(\mu_{\overline{Q}}^{\mathbf{r}}, \upsilon\Sigma_P^{\mathbf{r}}\right)$. Using Isserlis' theorem and the properties of Chi-squared and normal distributions, we solve the summary statistics A, B, C, $E_{\pi}\left(R_Q^{\mathbf{w}^d}\right)$, and $\sigma_{\pi}^2\left(R_Q^{\mathbf{w}^d}\right)$ in Corollary 2 for the optimal level of delegation, and the key covariance component in investors' optimal portfolio,

$$cov_{\pi}\left(\mu_{\overline{Q}}^{\mathbf{r}}, R_{\overline{Q}}^{\mathbf{w}^d}\right) = \frac{2\upsilon}{\gamma}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right).$$

The solution is summarized in the following proposition for comparative statics.

Proposition 4 (Comparative Statics) *Under the three assumptions, the investor's portfolio is given by*

$$\mathbf{w}^o = \left(\frac{1}{\gamma + \theta\upsilon}\right)\left(\Sigma_P^{\mathbf{r}}\right)^{-1}\left[\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right) - (\gamma+\theta)\left(\frac{\delta}{1-\delta}\right)\frac{2\upsilon}{\gamma}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)\right]. \tag{19}$$

The optimal delegation decision is

$$\delta = \frac{\frac{\upsilon}{\gamma}N - \psi + \left[1 - \frac{\gamma}{\gamma+\upsilon\theta}\left(\frac{2\upsilon(\theta+\gamma)}{\gamma} + 1\right)\right] R_{\overline{Q}}^{\mathbf{w}^d}}{\left(1 + 2\frac{\theta\upsilon}{\gamma}\right)\frac{\upsilon}{\gamma}N + \left[1 + 4\frac{\theta\upsilon}{\gamma} - \frac{\gamma}{\gamma+\upsilon\theta}\left(\frac{2\upsilon(\theta+\gamma)}{\gamma} + 1\right)^2\right] R_{\overline{Q}}^{\mathbf{w}^d}}, \tag{20}$$

where the expected return to the delegated portfolio under the average model $\overline{Q}$ is

$$R_{\overline{Q}}^{\mathbf{w}^d} = \left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)^T \left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right).[21] \tag{21}$$

We have the following results of comparative statics:

1 *The optimal level of delegation δ increases in N, the number of risky asset, and γ, the risk aversion: $\frac{\partial\delta}{\partial N} > 0$, $\frac{\partial\delta}{\partial\gamma} > 0$.*

2 *The optimal level of delegation δ decreases in θ, the ambiguity aversion, υ, the level of ambiguity, and ψ, the management free: $\frac{\partial\delta}{\partial\upsilon} < 0$, $\frac{\partial\delta}{\partial\theta} < 0$, $\frac{\partial\delta}{\partial\psi} < 0$.*

3 *Given the delegation level δ, $\mathbf{w}^o$ decreases in θ, the ambiguity aversion, υ, the level of ambiguity, and γ, the risk aversion: $\frac{\partial\mathbf{w}^o}{\partial\upsilon} < 0$, $\frac{\partial\mathbf{w}^o}{\partial\theta} < 0$, $\frac{\partial\mathbf{w}^o}{\partial\gamma} < 0$, given δ.*

4 *When, $N < \frac{1}{\upsilon}\left[\left(\gamma + \theta + \frac{\gamma}{2\upsilon}\right)\psi + (\theta-\gamma) R_{\overline{Q}}^{\mathbf{w}^d}\right]$, $\mathbf{w}^o \geq \mathbf{0}$ if and only if $\mu_{\overline{Q}}^{\mathbf{r}} \geq r_f\mathbf{1}$.*

[21] A simple calculation shows that the formula produces reasonable level of delegation. δ equals 49% under the following calibration: $N = 10$, $\gamma = 5$, $\theta = 1$, $R_{\overline{Q}}^{\mathbf{w}^d} = 0.04$, $\psi = 0.01$ and $\upsilon = 0.01$. δ increases to 99%, when N increases to 1000.

After applying the Isserlis' theorem to simplify δ and $\mathbf{w}^o$ (details in the Appendix), a new summary statistic N, the number of assets, shows up. Because across models, the expected delegation return, $R_Q^{\mathbf{w}^d} = \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1}\right)^T \mathbf{w}^d(Q) = \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1}\right)^T \left(\gamma \Sigma_P^{\mathbf{r}}\right)^{-1} \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1}\right)$, follows Chi-squared distribution (because investors' prior π is Gaussian), so N appears because it is the degree of freedom in the mean and variance formula of Chi-squared distributions. We provide detailed derivation in the Appendix.

Intuitively, as the number of risky assets increases, the fund manager's ability to construct efficient portfolios of a large set of assets is more valuable, so the delegation level increases. Higher risk aversion increases the wealth delegated to managers who construct efficient portfolios, because when risk aversion is high, being away from the frontiers significantly decreases the investor' utility.

Note that we can interpret N as the number of risk factors instead of primitive risky assets. Suppose there are infinite number of assets, whose returns are spanned by N risk factors and their own idiosyncratic shocks. By law of large numbers, the investor can always diversify away idiosyncratic shocks at zero cost no matter which probability model is true, as long as candidate probability measures are not point-mass. Effectively, the investor deals with N risk factors. More sources of risk motivates the investor to delegate more.

Holding constant N, delegation decreases in ambiguity aversion (θ) and the level of ambiguity (υ), because the need to hedge against delegation uncertainty is stronger. The benefit of delegation is that the δ fraction of wealth is allocated efficiently, but the more the investor delegates, the stronger the cross-model hedging motive, which which reduces benefits of delegation. A more uncertain environment tends to reduce delegation.

The comparative statics on investor's portfolio choice are derived given the optimal delegation. The investor becomes more conservative in holding risky assets, when facing more ambiguity, or under higher ambiguity aversion or risk aversion.[22]

In reality, most investors hold long positions. In the model, investors takes all long positions, if under their average model, the expected excess returns are non-negative ($\mu_Q^{\mathbf{r}} \geq r_f 1$). This result requires N to be lower than an upper bound. As historic data accumulates, υ, the estimation error, raising the upper bound of N. The upper bound of N is equal to 272 under the following calibration: $\gamma = 5$, $\theta = 1$, $R_Q^{\mathbf{w}^d} = 0.04$, $\psi = 0.01$ and $\upsilon = 0.01$. This

[22]Gollier (2011) investigated the comparative statics of more ambiguity aversion in a static two-asset portfolio problem.

number is likely to be larger than the number of systematic risk factors.

2.4 Application I: Cross-section asset pricing

We characterize the cross section of expected asset returns and their CAPM alpha. First, we show that when delegation is unavailable, our model produces results that nest key theoretical findings in the literature of asset pricing with ambiguity. Next, we show that adding delegation significantly changes the results. In contrast to the existing literature, the CAPM alpha (the "ambiguity premium"), does not disappear even when investors are not ambiguity-averse. Also, if we consider a sequence of economies with increasing levels of delegation all the way to 100%, the asset market equilibrium does not converge to CAPM. The key to these results is investors' hedging against delegation uncertainty. We solve these results under the general form of model uncertainty.

To characterize the equilibrium expected return, we define the market portfolio $\mathbf{m}$, which is the exogenous supply of risky assets. The market clearing condition equates the supply with the demand, which is the sum of investors' and managers' portfolios,

$$\mathbf{m} = \delta \mathbf{w}^d (P) + (1-\delta)\, \mathbf{w}^o. \tag{22}$$

Equilibrium without delegation. We first study the case without delegation. Recall that $\mathbf{w}_0^o$, the "zero-delegation portfolio", is investor's portfolio when delegation is unavailable,

$$\mathbf{w}_0^o = \left(\gamma \Sigma_{\overline{Q}}^{\mathbf{r}} + \theta \Sigma_{\pi}^{\mu_Q^{\mathbf{r}}}\right)^{-1} \left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right) \tag{23}$$

When $\delta = 0$, substituting the market clearing condition, $\mathbf{m} = \mathbf{w}_0^o$, into Equation 23 and multiplying both sides by $\left(\gamma \Sigma_{\overline{Q}}^{\mathbf{r}} + \theta \Sigma_{\pi}^{\mu_Q^{\mathbf{r}}}\right)$, we have

$$\mu_{\overline{Q}}^{\mathbf{r}} - r_f \mathbf{1} = \left(\gamma \Sigma_{\overline{Q}}^{\mathbf{r}} + \theta \Sigma_{\pi}^{\mu_Q^{\mathbf{r}}}\right) \mathbf{m}. \tag{24}$$

Note that $\Sigma_{\overline{Q}}^{\mathbf{r}} \mathbf{m}$ is simply the vector of covariance under $\overline{Q}$ between asset returns and the market return, and $\Sigma_{\pi}^{\mu_Q^{\mathbf{r}}} \mathbf{m}$ records the covariance under π between *expected* asset returns and the expected market return. If investors' average model is true, i.e., $\overline{Q} = P$, the left-hand side is the assets' expected excess returns under the true probability measure, and the right-hand side is decomposed into two covariance terms.

Proposition 5 (Ambiguity premium without delegation) *When delegation is unavailable ($\delta = 0$), the equilibrium expected excess returns of risky assets are*

$$\mu_P^{\mathbf{r}} - r_f \mathbf{1} = \lambda_{\mathbf{m}} \boldsymbol{\beta}_{\mathbf{r},\mathbf{m}}^P + \lambda_{\mathbf{w}_0^o} \boldsymbol{\beta}_{\mu_Q^{\mathbf{r}},\mathbf{m}}^{\pi}, \tag{25}$$

if investors' average model is the true model, i.e., $\overline{Q} = P$, where we define

- *market price of risk, $\lambda_{\mathbf{m}} = \gamma \sigma_P^2 (R^{\mathbf{m}})$, the risk beta, $\boldsymbol{\beta}_{\mathbf{r},\mathbf{m}}^P = \frac{cov_P(\mathbf{r}, R^{\mathbf{m}})}{\sigma_P^2(R^{\mathbf{m}})}$,*
- *market price of ambiguity, $\lambda_{\mathbf{w}_0^o} = \theta \sigma_\pi^2 \left(R_Q^{\mathbf{m}}\right)$, the ambiguity beta, $\boldsymbol{\beta}_{\mu_Q^{\mathbf{r}},\mathbf{m}}^{\pi} = \frac{cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{m}}\right)}{\sigma_\pi^2\left(R_Q^{\mathbf{m}}\right)}$.*

Equation (25) decompose the expected excess return into two components. When investors are the only market participants, the expected excess returns compensate them for both their risk exposure and ambiguity exposure. The first component $\boldsymbol{\lambda}_{\mathbf{m}} \boldsymbol{\beta}_{\mathbf{r},\mathbf{m}}^P$ is exactly the standard CAPM beta multiplied by the market price of risk. The second term $\lambda_{\mathbf{w}_0^o} \boldsymbol{\beta}_{\mu_Q^{\mathbf{r}},\mathbf{m}}^{\pi}$ is the product of the ambiguity beta and price of ambiguity.

The ambiguity beta measures the cross-model comovement between the expected asset returns and the expected market return (i.e. the return of zero-delegation portfolio). If asset i's expected return comoves with the expected market return across models (i.e. $\boldsymbol{\beta}_{\mu_Q^{\mathbf{r}_i},\mathbf{m}}^{\pi} > 0$), the asset must deliver a higher average return through $\lambda_{\mathbf{w}_0^o} \boldsymbol{\beta}_{\mu_Q^{\mathbf{r}_i},\mathbf{m}}^{\pi} > 0$ in equilibrium. If asset i's expected return moves against the expected market return (i.e. $\boldsymbol{\beta}_{\mu_Q^{\mathbf{r}_i},\mathbf{m}}^{\pi} < 0$), then it serves as hedge against model uncertainty from investor's perspective, and thus, it affords a discount in the average return via $\lambda_{\mathbf{w}_0^o} \boldsymbol{\beta}_{\mu_Q^{\mathbf{r}_i},\mathbf{m}}^{\pi} < 0$.

The assumption of $\overline{Q} = P$ is important. Investors face model uncertainty, so they cannot evaluate the expected returns of risky assets under the true model P. Instead, they examine the expected returns by averaging over candidates models, i.e., $\mu_{\overline{Q}}^{\mathbf{r}}$, and accordingly, expected returns under $\overline{Q}$ reflect investors' demand for risk and ambiguity compensation. Only under the assumption that $\overline{Q} = P$, do investors' expected returns $\mu_{\overline{Q}}^{\mathbf{r}}$ coincide with the expected returns under the true model $\mu_P^{\mathbf{r}}$, which are observed by econometricians, and thus, can we solve $\mu_P^{\mathbf{r}}$ using the portfolio optimality condition (substituting out $\mathbf{w}^o$ with $\mathbf{m}$).

Ambiguity generates CAPM alpha as in Maccheroni, Marinacci, and Ruffino (2013). They analyze a special case of two assets where one asset is pure risk (whose distribution is known) while the other asset's return is ambiguous. Using the constrained-robust approach, Kogan and Wang (2003) derive the similar two-factor structure of equilibrium expected returns. In those models and here, if we shut down ambiguity aversion ($\theta = 0$), the price of ambiguity, $\lambda_{\mathbf{w}_0^o} = \theta\sigma_\pi^2\left(R_Q^{\mathbf{w}_0^o}\right)$, is zero, and the model degenerates to CAPM.

Corollary 3 (CAPM without delegation) *When delegation is unavailable ($\delta = 0$), if investors are ambiguity-neutral ($\theta = 0$), the equilibrium excess returns of risky assets are*

$$\mu_P^{\mathbf{r}} - r_f\mathbf{1} = \lambda_{\mathbf{m}}\boldsymbol{\beta}_{\mathbf{r},\mathbf{m}}^P, \tag{26}$$

if investors' average model is the true model, i.e., $\overline{Q} = P$.

If the investor is ambiguity-neutral, the investor's utility function can be written as

$$V(r_\omega) = \int_\Delta\int_{\omega\in\Omega} u(r_\omega)\, dQ(\omega)\, d\pi(Q) = \int_{\omega\in\Omega} u(r_\omega)\left[\int_\Delta dQ(\omega)\, d\pi(Q)\right] = \int_{\omega\in\Omega} u(r_\omega)\, d\overline{Q}(\omega)$$

which is simply the expected utility given the average probability model $\overline{Q}$. Our quadratic approximation becomes the standard mean-variance utility as shown in Corollary 1, so if $\overline{Q} = P$, we rediscover CAPM. It is critical that $u(r)$ can be taken out of the integral operator $\int_\Delta$, because $u(r)$, or equivalently r, only depends on the state ω, but not on the model Q. This is in turn because delegation is unavailable, so investors' wealth is not model-contingent. Next, we show that when delegation is available, the equilibrium deviates from CAPM even when investors are ambiguity-neutral.

Equilibrium with delegation. When delegation is available, the market portfolio is equal to a mixture of managers' portfolio and investors' portfolio, i.e., $\mathbf{m} = \delta\mathbf{w}^d(P) + (1-\delta)\mathbf{w}^o$. We arrange the fund manager's portfolio, $\mathbf{w}^d(P) = (\gamma\Sigma_P^{\mathbf{r}})^{-1}(\mu_P^{\mathbf{r}} - r_f\mathbf{1})$, under the true probability distribution P, and arrive at the following expression of expected excess returns:

$$\mu_P^{\mathbf{r}} - r_f\mathbf{1} = (\gamma\Sigma_P^{\mathbf{r}})\,\mathbf{w}^d(P).^{23} \tag{27}$$

Substituting the rearranged market clearing condition, $\mathbf{w}^d(P) = \frac{1}{\delta}\mathbf{m} - \left(\frac{1-\delta}{\delta}\right)\mathbf{w}^o$, into the

[23] Note that because $\mu_P^{\mathbf{r}}$ already shows up in managers' portfolio, we do not need to assume $\overline{Q} = P$ to solve the equilibrium expected returns as we did for the case without delegation.

equation above, we have

$$\mu_P^{\mathbf{r}} - r_f \mathbf{1} = \underbrace{\frac{1}{\delta}\gamma\Sigma_P^{\mathbf{r}}\mathbf{m}}_{\boldsymbol{\beta}_{\mathbf{r},\mathbf{m}}^P \lambda_\delta} \underbrace{- \left(\frac{1-\delta}{\delta}\right)\gamma\Sigma_P^{\mathbf{r}}\mathbf{w}^o}_{\boldsymbol{\alpha}}. \tag{28}$$

Using the definition of market beta in Proposition (5), we may rewrite the first term on the right-hand side as the product of assets' market beta and the price of market risk, λ_δ.

In contrast to the equilibrium without delegation, the price of market risk, $\lambda_\delta = \frac{\gamma}{\delta}\sigma_P^2(R^{\mathbf{m}})$, decreases in the level of delegation δ. This property is in line with the concurrence of a growing asset management industry and a declining equity premium in the U.S. market (documented by Jagannathan, McGrattan, and Scherbina (2001) and Lettau, Ludvigson, and Wachter (2008) among others). A declining market price of risk in response to a rising level of delegation suggests that the security market line becomes flatter as more money is poured into the professional asset management industry.

The deviation from CAPM, $\boldsymbol{\alpha}$, is due to the second term on the right-hand side, which depends on the investors' portfolio (solved in Equation (2) and reproduced below),

$$\mathbf{w}^o = \left(\gamma\Sigma_{\overline{Q}}^{\mathbf{r}} + \theta\Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)^{-1}\left[\underbrace{\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)}_{\text{“sentiment”}} \underbrace{-(\theta+\gamma)\left(\frac{\delta}{1-\delta}\right)cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)}_{\text{delegation hedging demand}}\right],$$

that combines “sentiment” (the expected asset returns under the average model, or *average belief*) and the uncertainty hedging demand that depends on the cross-model covariance between the expected asset returns and the expected delegation return, and is scaled by both risk ($\Sigma_{\overline{Q}}^{\mathbf{r}}$) and model uncertainty ($\Sigma_\pi^{\mu_Q^{\mathbf{r}}}$). Substituting this expression of $\mathbf{w}^o$ into the $\boldsymbol{\alpha}$ component of Equation (28), we have

$$\boldsymbol{\alpha} = -\gamma\Sigma_P^{\mathbf{r}}\left(\gamma\Sigma_{\overline{Q}}^{\mathbf{r}} + \theta\Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)^{-1}\left[\underbrace{\left(\frac{1-\delta}{\delta}\right)\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)}_{\text{from average belief}} - \underbrace{(\theta+\gamma)\,cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)}_{\text{from delegation hedging}}\right] \tag{29}$$

The sentiment component of $\boldsymbol{\alpha}$ captures investors' belief (averaged over models). It eventually disappears if the level of delegation approaches 100% (i.e., $\delta \to 1$). This is consistent with Corollary 3 because this component is exactly the full $\boldsymbol{\alpha}$ when delegation is unavailable. Such deviation from CAPM shrinks as the market is increasingly dominated by rational-expectation managers who know the true probability model and allocate wealth on the efficient frontier.

The other component $\boldsymbol{\alpha}$, which is from investors uncertainty hedging demand, is immune to the rise of delegation level, and in particular, it does not disappear even when delegation approaches 100%. This hedging demand arises from delegation. In fact, as δ approaches 100%, the hedging demand becomes increasingly significant in magnitude as shown by the multiplier $\left(\frac{\delta}{1-\delta}\right)$ in front of $cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)$ in investors' portfolio. Therefore, even if investors manage less wealth when δ increases, the hedging incentive is stronger *per unit* of retained wealth. Intuitively, the more wealth is delegated to managers, the more investors want to hedge against the cross-model variation of delegation return. When investors' portfolio enter into the expression of expected asset returns in Equation (28), this multiplier exactly offsets $\left(\frac{1-\delta}{\delta}\right)$, the ratio of retained-to-delegated wealth, sustaining the $\boldsymbol{\alpha}$ so that the economy does not converge to CAPM even if δ approaches 100%.

Interestingly, we may set θ, the ambiguity aversion parameter, equal to zero, but the component of $\boldsymbol{\alpha}$ from delegation hedging still exists and approaches $\gamma cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)$ as δ approaches 100%. This property distinguishes our model from existing models of asset pricing with ambiguity, in which $\boldsymbol{\alpha}$ disappears if investors are no longer ambiguity-averse (e.g., Kogan and Wang (2003)). The key is delegation, as we have shown in Corollary 3 that when delegation is unavailable, $\boldsymbol{\alpha}$ disappears when $\theta = 0$. We can understand the intuition behind our result by inspecting an ambiguity-neutral investor's utility function as we did for the case without delegation, but now notice that the return on wealth is both state-dependent and, through delegation, model-dependent.

$$V\left(r_{\omega,Q}\right) = \int_{Q\in\Delta}\int_{\omega\in\Omega} u\left(r_{\omega,Q}\right) dQ\left(\omega\right) d\pi\left(Q\right),$$

where a distribution of states of the world, $Q(\omega)$, can be viewed as a distribution conditional on Q. Due to delegation, an ambiguity-neutral investor cannot perform Bayesian model averaging and operates under the average probability $\overline{Q}$, but instead, has to deal with the *joint* uncertainty of state and model. Therefore, the cross-model covariance between the expected asset returns and the expected delegation return still appears in investors' portfolio choice, and the equilibrium expected asset returns, even if $\theta = 0$.

Proposition 6 summarizes our results so far.

Proposition 6 (Ambiguity premium with delegation) *The equilibrium expected excess returns of risky assets are given by*

$$\mu_P^{\mathbf{r}} - r_f \mathbf{1} = \lambda_\delta \boldsymbol{\beta}_{\mathbf{r},\mathbf{m}}^P + \boldsymbol{\alpha}. \tag{30}$$

The market betas, $\boldsymbol{\beta}_{\mathbf{r},\mathbf{m}}^P$, are defined as in Proposition 5. The price of market risk, $\lambda_\delta = \frac{\gamma}{\delta}\sigma_P^2(R^{\mathbf{m}})$, suggesting that the security market line become flatter as delegation increases.

The CAPM $\boldsymbol{\alpha}$ is given by Equation (29), which depends investors' sentiment, $\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right)$ (the average belief), and $cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)$, the cross-model covariance between the assets' expected returns and the expected delegation return.

When δ approaches 100% (for example because the management fee declines or the number of assets increases as in Proposition 4), $\boldsymbol{\alpha}$ does not converge to zero. Its limit is $\gamma\Sigma_P^{\mathbf{r}}\left(\gamma\Sigma_{\overline{Q}}^{\mathbf{r}} + \theta\Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)^{-1}(\theta+\gamma)\, cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)$, a linear transformation of $cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)$.

Even if investors are not ambiguity-averse ($\theta = 0$), $\boldsymbol{\alpha}$ still exists, and when δ approaches 100%, it approaches $\gamma cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)$.

Note that when δ is precisely equal to 100%, we have $\mathbf{m} = \mathbf{w}^d(P)$, and rediscover CAPM,

$$\mu_P^{\mathbf{r}} - r_f \mathbf{1} = \boldsymbol{\beta}_{\mathbf{r},\mathbf{m}}^P \lambda_{\mathbf{m}}, \tag{31}$$

where $\lambda_{\mathbf{m}} = R_P^{\mathbf{w}^d} = R_P^{\mathbf{m}}$. However, as long as $\delta < 100\%$, investors need to allocate their retained wealth under ambiguity. The more they delegate, the stronger they hedge against delegation uncertainty in their portfolio choice (Equation (12)). As shown in Proposition 6, what generates alpha is this hedging demand. Therefore, even if the total amount of retained wealth declines as δ increases, the hedging demand increases per unit of retained wealth, and thus, sustains the alpha.

Corollary 4 (Equilibrium discontinuity with delegation) *When δ approaches 100% (for example because the management fee declines or the number of assets increases as in Proposition 4), the model equilibrium does not converge to the CAPM equilibrium. However,* *when $\delta = 100\%$, the model produces the CAPM equilibrium. Therefore, there exists an equilibrium discontinuity in the limit.*

In the past few decades, asset management industry has grown dramatically, especially in the area of quantitative investment that often targets alpha already identified in the academic literature. Many have argued that investment strategies' alpha shrinks as arbitrage capital increases. Yet many strategies survive, and together, they constitute a rich set of "anomalies" in asset pricing. In our model, as shown in Proposition 4, the asset management industry grows (i.e, δ increases) for several reasons. The number of systematic risk factors, N, may have increased due to technological changes or globalization. The management fee, ψ, may have decreased thanks to increasing competition and more efficient data processing. As δ becomes higher, an increasing share of the asset market is taken by managers who hold the frontier portfolio. Will the equilibrium converge to CAPM? The answer is no.

Empirically, we can observe the growth of professional asset management, but CAPM alpha never disappears for certain assets or investment strategies. It is difficult for asset pricing models to rationalize such "anomalies". It is even more difficult to reconcile the robust alpha of several anomalies in a period of increasing arbitrage capital. Our model offers a new perspective to understand such phenomena.

2.5 Application II: Delegation and fund performance

Evidence on the mediocre fund performance suggests that investors are better off not delegating and holding indices instead (reviewed by French (2008)). This poses a challenge to understand the growth of professional asset management. Our model shifts the focus from ex post performance to ex ante welfare. Performance measurement assumes "large sample" and investors have rational expectation (i.e., econometricians' belief). In reality, investors face model uncertainty. In our model, managers allocate the delegated wealth efficiently for each model, but through delegation, investors' wealth becomes model-contingent. When choosing the optimal level of delegation, the trade-off is now between *within-model* allocation efficiency and *cross-model* delegation uncertainty. Next, we characterize conditions under which delegation arises in spite of underperformance relative to the market index or the negative CAPM alpha delivered by fund managers.

Fund underperforming the market. Let us consider investing in the market index, and compare the expected delegation return and the market return under the simplified structure of ambiguity. Substituting the investor's portfolio (equation (19)) into the expected market excess return, we solve the expected market return under the true probability distribution:

$$\begin{aligned} R_P^{\mathbf{m}} &= \delta R_P^{\mathbf{w}^d} + (1-\delta) R_P^{\mathbf{w}^o} \\ &= \left(\mu_P^{\mathbf{r}} - r_f \mathbf{1}\right)^T \left(\gamma \Sigma_P^{\mathbf{r}}\right)^{-1} \left[\left(\mu_P^{\mathbf{r}} - r_f \mathbf{1}\right)\delta + \left(\mu_{\underline{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right)\left(\frac{(1-\delta)\gamma - \delta 2\upsilon(\theta+\gamma)}{\gamma + \upsilon\theta}\right)\right]. \end{aligned}$$

The expected excess return of the fund manager's portfolio is

$$R_P^{\mathbf{w}^d} = \left(\mu_P^{\mathbf{r}} - r_f \mathbf{1}\right)^T \left(\gamma \Sigma_P^{\mathbf{r}}\right)^{-1} \left(\mu_P^{\mathbf{r}} - r_f \mathbf{1}\right)$$

The difference between the expected fund return and the expected market return, $R_P^{\mathbf{w}^d} - R_P^{\mathbf{m}}$, is equal to

$$(1-\delta)\left(\mu_P^{\mathbf{r}} - r_f \mathbf{1}\right)^T \left(\gamma \Sigma_P^{\mathbf{r}}\right)^{-1} \left[\left(\mu_P^{\mathbf{r}} - r_f \mathbf{1}\right) - \left(\mu_{\underline{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right)\left(\frac{\gamma - \left(\frac{\delta}{1-\delta}\right) 2\upsilon(\theta+\gamma)}{\gamma + \upsilon\theta}\right)\right], \quad (32)$$

which is also the average performance difference in a *large* sample.

Proposition 7 (Delegation and underperformance) *Under the three simplification assumptions, fund managers underperform the market if*

$$\sum_{i=1}^{N} \mathbf{w}_i^d(P)\left(\mu_P^{\mathbf{r}_i} - r_f\right) < \varkappa \sum_{i=1}^{N} \mathbf{w}_i^d(P)\left(\mu_{\underline{Q}}^{\mathbf{r}_i} - r_f \mathbf{1}\right),$$

where $\mathbf{w}_i^d(P)$ *is the fund managers' portfolio weight on asset* i *under the true probability* P*, and*

$$\varkappa = \frac{\gamma - \left(\frac{\delta}{1-\delta}\right) 2\upsilon(\theta+\gamma)}{\gamma + \upsilon\theta}, \quad (33)$$

which increases in θ *and* υ *and decreases in* γ*.*

Whether the fund managers underperform or outperform the market depends on the comparison between the weighted-average of assets' expected returns under true model and the weighted average of assets' expected returns under the investors' average model (scaled by $\varkappa$). Because investors also participate in the market, fund managers' performance depends on their relative aggression in risk- and ambiguity-taking. For example, if investors have in mind a high-return market (i.e. high $\mu_Q^{\mathbf{r}}$), then they can be more aggressive and earn a higher expected return than fund managers by taking on more exposure to risk and ambiguity.

Therefore, in our model, delegation can arise in spite of managers' underperformance relative to the market. Investors do not know the true probability distribution, so they cannot evaluate fund performance under rational expectation and choose between funds or the market index. Note that we do not impose any restriction on investors' portfolio choice, so holding the market portfolio is certainly within investors' opportunity set. But

Delegation without alpha. Another commonly used performance metric is "alpha" (Jensen (1968)). It is defined as the residual average from regressing fund return on the market return. Let us assume that $\mu_Q^{\mathbf{r}} = \mu_P^{\mathbf{r}}$. In this case, investors' portfolio is proportional to fund managers' portfolio (and the market portfolio) under the simplified ambiguity (see Equation (19) and below):

$$\mathbf{w}^o = \left(\frac{1 - (\gamma + \theta) \left(\frac{\delta}{1-\delta} \right) \frac{2\upsilon}{\gamma}}{\gamma + \theta \upsilon} \right) \left(\Sigma_P^{\mathbf{r}} \right)^{-1} \left(\mu_P^{\mathbf{r}} - r_f \mathbf{1} \right). \tag{34}$$

Therefore, CAPM holds. A regression of managers' return on the market return shows exactly zero alpha in a large sample, so after fees, investors receive negative alpha from delegation. Moreover, fund managers hold the market portfolio up to a scaling factor, as some have documented in the empirical literature (e.g., Lewellen (2011)).

Proposition 8 (Delegation and negative alpha) *Under the simplified model uncertainty and given $\mu_Q^{\mathbf{r}} = \mu_P^{\mathbf{r}}$, the delegated portfolio delivers zero gross alpha and negative alpha after fees, and it is proportional to the market portfolio.*

Why investors invest a significant share of wealth in actively managed funds, in spite of their underperformance and negative alpha after fees. This paper argues that under ambiguity, they choose to delegate in order to improve ex ante welfare. When choosing the optimal level of delegation, the trade-off is between within-model allocation efficiency and cross-model delegation uncertainty. Such focus on ex ante welfare echoes the observation by Gennaioli, Shleifer, and Vishny (2015).

Another interesting implication is that even if fund manager possesses superior knowledge and knows the true model, this may not help them to generate "market risk-adjusted return". This result challenges the traditional approach of fund performance measurement: an asset management firm could be "active" in acquiring the knowledge of true return distribution, but this effort is not likely to be compensated if we only look at alpha.

3 Evidence

In this section, we provide supporting evidence for our modeling assumptions and theoretical results. We use data from the U.S. stock market, and consider investors' asset set spanned by well-studied factors. As discussed previously, idiosyncratic risks can be diversified away, so they should not matter for investors' evaluation of risk and ambiguity, and thus, we choose factors instead of individual stocks as the basis for investment opportunity set. Below we provide a preview of empirical results.

In the model with simplified ambiguity (under Assumption 1 – 3), the alphas of assets (factors in our empirical setting) are proportional to investors' "sentiment", i.e., $\mu_Q^{\mathbf{r}} - r_f \mathbf{1}$, the expected excess returns of assets under the average model (see Equation (**??**)). Moreover, investors' portfolio weights on assets are also proportional to sentiment (see Equation (19)). Therefore, we can use assets' ownership by investors, or by fund managers, as proxy for assets' alpha and expected returns.

Indeed, we find that the current ownership by fund managers predicts future factor returns. Therefore, fund managers exhibit superior knowledge of the expected factor returns, and they perform factor timing. This finding is line with our model assumption that managers have advantage in knowing the return distribution.

Next, we examine the model prediction that as the delegation level grows, CAPM alpha does not disappear for a set of factors. We plot the faction of wealth in the U.S. stock market managed in delegated portfolios, and the rolling-window CAPM alpha of a set factors with high fund ownership (i.e., those tend to outperform according to our previous findings). The delegation level exhibits a strong upward trend, but in spite of this, the alpha of our factor portfolio fluctuates and stays above zero consistently.

Finally, we simulate investors' ambiguity by fitting a latent factor model to the returns of commonly used size and book-to-market sorted portfolios. Specifically, the model features time-varying covariance matrix to be consistent with the literature on volatility persistence (reviewd by Andersen, Bollerslev, Christoffersen, and Diebold (2006)). Ambiguity, measured by the Bayesian posterior uncertainty, is directly plugged into the optimal delegation level in the model. The model-implied delegation has a 19% correlation with the detrended data.

3.1 Data sources and variable construction

Asset space: factors. We consider the most well-studied stock-market factors in the empirical asset pricing literature. The factors can be divided into two categories: accounting-based and return-based. Accounting-based factors include value ("HML"), accruals ("ACR"), investment ("CMA"), profitability ("RMW"), and net issuance ("NI"). Return-based factors include momentum ("MOM"), short-term reversal ("STR"), long-term reversal ("LTR"), betting-against-beta ("BAB"), idiosyncratic volatility ("IVOL"), and total volatility ("TVOL").

To construct each factor, we use monthly and daily returns data of stocks listed on NYSE, AMEX, and Nasdaq from the Center for Research in Securities Prices (CRSP). We include ordinary common shares (share codes 10 and 11) and adjust delisting by using CRSP delisting returns. We obtain accounting data from annual COMPUSTAT files to compute firm characteristics. We follow the standard convention and lag accounting information by six months (Fama and French (1993)). If a firm's fiscal year ends in December in year t, we assume that this information is available to investors at the end of June in year $t+1$.

We construct each factor in the typical HML-like fashion by independently sorting stocks into six portfolios by size ("ME") and the factor characteristic. We use standard NYSE breakpoints – median for size, and 30th and 70th percentiles for the factor characteristic. We compute value-weighted returns and other statistics of the six portfolios. A factor's return is the value-weighted average return of the two high-characteristic portfolios minus that of the two low-characteristic portfolios. We rebalance accounting-based factors annually at the end of each June, and rebalance the return-based factors monthly.

Fund ownership: δ in data. We use quarterly institutional ownership data from Thompson Financial CDA/Spectrum database from 1980Q1 to 2017Q4. Mutual fund characteristics (e.g., investment objectives) are obtained from the CRSP survivorship-bias-free mutual fund database. We apply standard filters to holdings data following the literature: (1) we pick the first vintage date ("FDATE") for each fund-report date (FUNDNO-RDATE) pair to avoid stale information; (2) we adjust shares held by a fund for stock splits to account for corporate events that happen between report date ("RDATE") and vintage date ("FDATE").

We select funds focusing on the U.S. stock market by excluding those with investment objective codes ("IOC") of International, Municipal Bonds, Bond & Preferred, and Balanced. For the main results, we map institutional investors to managers in our model. As a robustness check, we further narrow down the definition of institutional investors to *active* domestic equity funds by utilizing investment objective codes from CRSP, Lipper, Strategic Insight, and Wiesenberger. The results using this narrower definition of managers are very similar to our main results (available upon request).

We calculate managers' ownership by summing up the stock holdings of institutional investors for each stock in each quarter. Stocks that are on listed in CRSP, but without any reported institutional holdings, are assumed to have zero fund ownership. Table 1 reports summary statistics of monthly returns and quarterly fund ownership for all factors.

Fund ownership at factor level. Our model is built upon the assumption that asset managers have superior knowledge of factor return distribution. A particular implication is that the variation of $\mathbf{w}^d$, i.e., the portfolio rebalancing across factors by fund managers, should predict future factor returns – asset managers have superior information on the first moment of factor returns. Ideally, we would like to treat each factor as an asset and compute the weight for each factor as the fraction of total dollar amount invested by funds. However, factors are comprised of numerous stocks and different factors have overlapping stock compositions. For example, stock A could be in the long leg of value and short leg of momentum. The exact dollar amount of stock A attributed to each factor cannot be exactly identified, which complicates our portfolio weight calculation.

Instead of calculating the exact weights of factors in fund portfolio, we calculate the relative over/underweight of each factor. Specifically, we measure the professional asset managers' allocation to each factor by the spread of institutional ownership ("$INST$") between the long leg and short leg:

$$INST_{i,t} = INST_{i,t}^{long} - INST_{i,t}^{short} \tag{35}$$

where $INST_{i,t}^{j}, j = \{long, short\}$ is the value-weighted average of the institutional ownership of all constituent stocks in long/short leg of factor i. The intuition is simple. If managers have superior knowledge of the true return distribution, when they overweight certain factors, the subsequent performance of these factors shall be stronger on average. Therefore, in the following, we will use $INST_{i,t}$ to forecast factor i's future return.

3.2 Asset pricing implications

Factor timing: a parametric test. Using asset managers' allocation to factors ($INST$), we test whether they have superior knowledge of return distributions. Specifically, we estimate the following predictive regression: for factor i,

$$R_{i,t,t+3} = \alpha + \beta \cdot INST_{i,t} + \gamma \cdot X_{i,t} + \varepsilon_{i,t,t+3} \tag{36}$$

where $i = \{HML, ACR, CMA, RMW, NI, MOM, STR, LTR, BAB, IVOL, TVOL\}$, $R_{i,t,t+3}$ is the return next quarter (i.e., month t to $t+3$), and $X_{i,t}$ includes control variables such as factor volatility that may also predict factor returns (Moreira and Muir (2017)). We use the next-quarter return because institutional ownership data is available quarterly for individual stocks. Note that $INST$ at factor level varies every month due to the monthly rebalancing of value-weighted factor portfolios. Therefore, our estimation is at monthly level but with overlapping left-hand side variables. Our hypothesis is that a factor will deliver higher return in the future if its manager ownership $INST$ is higher now.

To increase statistical power, we pool all factors together to estimate a panel predictive regression. In Table 2 Panel A, we report the regression results using pooled OLS and various fixed effect models. $RV_{i,t}$ is the realized volatility of factor i estimated using previous 36 months of factor returns. Standard errors are double-clustered by factor and quarter.

As typical in the literature of return predictability, we address the concern over biased standard errors due to overlapping observations. Specifically, we follow the suggestion of Hodrick (1992) and run the following "reverse" regression to test the factor return predictability of $INST$ at three-month horizon.

$$3 \times R_{i,t+1m} = \alpha + \beta \left(\frac{1}{3} \sum_{j=0}^{2} INST_{i,t-j} \right) + \gamma \cdot X_{i,t} + \varepsilon_{i,t+1} \tag{37}$$

On the left-hand side is $R_{i,t+1m}$, the future one-month return multiplied by 3 so that it is comparable in magnitude with quarterly returns. Results are reported in Table 2 Panel B.

Our key prediction is confirmed in all specifications. In the both panels, the predictive coefficient of $INST$ is positive and significant, robust to alternative standard errors and various fixed effects. The coefficients estimated using panel regressions and Hodrick reverse regressions are very close. Moreover, the predictability we document is economically meaningful. For example, the coefficient 0.31 in the first column of Panel B implies that, when the institutional ownership of one factor rises by one standard deviation, future factor return increase by 44 bps in the following quarter (1.76% annualized). Given the average annual factor return of 3.31% in our sample, an one standard-deviation change in $INST$ is associated with 53% increase of expected factor return. The evidence of factor timing by fund managers lends substantial support to our model setup, the key assumption that asset managers possess superior knowledge of return distribution.

Our findings are interesting even independent from the theoretical setup, and add to the empirical literature on institutional ownership and asset return predictability. As documented by Nagel (2005), at stock level, returns are more predictable (by firm characteristics the cross section) when institutional ownership is low. Here, we find that at factor level, institutional ownership forecasts future factor returns.

Factor timing: nonparametric test. We also implement a trading strategy that exploits the information advantage of asset managers, which is a nonparametric test of our model setup. The strategy are formed as follows. At the end of each quarter, we rank all factors based on their $INST$. We long the top 4 factors and short the bottom 4 factors for the next quarter, weighing each factor equally. For comparison, we also form an "M" portfolio by equally weighing the factors with medium $INST$. The portfolio is rebalanced quarterly.

The performances of high $INST$ factors, low $INST$ factors, and that of long-short factor portfolio are plotted in Figure 1 (cumulative returns) and Figure 2 (rolling average returns). Factors with high fund ownership consistently outperform factors with low fund ownership since 1991. The fact that this pattern only started to appear in the 1990s suggests that asset management industry may benefit from the exploding research efforts devoted to stock-market factors in the academia, more data sources (the big data era), and the developments of data processing techniques, including financial econometrics in the 1990s.

So far, we have only focused on the first moment of factor returns. In Table 3, we report various moments and statistics of returns of factor portfolios sorted by fund ownership. Factors with high fund ownership exhibit higher mean return, lower volatility, and smaller skewness. These statistics all vary monotonically in fund ownership, suggesting that asset managers tend to invest in a set of factors with a desirable statistical profile (e.g., higher Sharpe ratio). Managers also tend to hold stocks with higher kurtosis relative to the rest of the market, which seems to imply that ambiguity investors refrain from factors with more extreme returns while asset managers are more willing to take on such exposure likely due to their confidence in gauging return distribution.

Alpha and the growth of professional asset management. In Corollary **??**, we show that even if the level of delegation approaches 100%, the equilibrium does not converge to CAPM. There exists a set of assets (factors in our empirical context) whose CAPM alpha is always non-zero. In Figure 3, we plot the aggregate fund ownership (right Y-axis) and the 60-month rolling-window estimate of CAPM alpha of the portfolio of high $INST$ factors. Given our previous results on factor timing, high $INST$ factors are more likely to exhibit higher future returns and CAPM alpha given that they are selected by asset managers.

Fund ownership exhibits a steady linear trend upward, rising from less than 5% in the 1980s to more than 20% recently. During this period, the alpha also trended up, from negative 40bps (monthly) to positive 60 bps (monthly) with occasional decline. But overall, there is no evidence that a growing asset management sector is associated with declining alpha or convergence to a CAPM economy. We also plot the 60-month rolling CAPM alpha of the long-short factor portfolio in Figure 4, and find similar patterns, which shows that results in Figure 3 is not restricted to a particular combination of factors

Corollary **??** also offers a decomposition of CAPM alpha into υ, the level of model uncertainty, and the expected asset returns under investors' average model ("optimism"). Alpha is higher when investors face more ambiguity or are more optimistic. In both Figure 3 and 4, we see that during an economic cycle (from boom to recession), alpha rises and then declines, suggesting these two forces dominate alternatively. In the early stage, optimism increases alpha, and as a boom prolongs, the declining ambiguity decreases alpha.

3.3 Model-implied fund ownership

The model has direct implication the optimal delegation level δ. To compare the dynamics of model-implied δ to data, we plot optimal delegation δ under the simulated ambiguity against the detrended empirical counterpart in Figure 5. We detrend because the rise of fund ownership may be due to technological progress, the evolution of stock market composition, and increasing specialization of labor that are outside of our model. Though the scales are different, the dynamics of model-implied and empirical δ are reasonably correlated. The correlations are 0.19 and 0.14 respectively with linearly detrended and HP-filtered empirical series. Below, we lay out the details on how to calculate model-implied δ.

Parameters and assets. The risk aversion γ is set to 2, and ambiguity aversion θ is set to 8.864. Both are chosen by Ju and Miao (2012) to match the risk-free rate and the equity premium under smooth ambiguity averse preference. The management fee ψ is 1%, in line with the asset management cost in the U.S. equity market (French (2008)). The risk-free rate is the one-month Treasury-bill rate. Returns of risky assets are monthly returns of the six size and book-to-market sorted portfolios from Kenneth French's website.

Ambiguity Structure. The investor holds the belief that asset returns are drawn from a normal distribution $N(\theta, \Sigma_t^{\mathbf{r}})$ with constant mean θ and time-varing covariance matrix $\Sigma^{\mathbf{r}_t}$.[24] The covariance matrix is decomposed into a time-invariant idiosyncratic part Ω, and a time-varing part BH_tB^T, where B is a constant matrix and H_t is a K-dimension diagonal matrix $diag\left(\{h_{k,t}\}_{k=1}^{K}\right)$ whose elements follow log-AR(1) process with i.i.d. normal shocks:

$$\log(h_{k,t}) = \alpha_k + \delta_k \log(h_{k,t-1}) + \sigma_k^{\upsilon} \upsilon_{k,t}, \;\; \upsilon_{k,t} \sim i.i.d.N(0,1) \tag{38}$$

This is the dynamic factor model of multivariate stochastic volatility studied by Jacquier, Polson, and Rossi (1999) and Aguilar and West (2000).

Therefore, each return model, $N(\theta, \Sigma^{\mathbf{r}_t}) \in \Delta$, is indexed by the values of parameters $(\alpha_k,\, \delta_k,\, \sigma_k^{\upsilon})$ and latent states $(h_{k,t})$. The uncertainty in these quantities spans the representative investor's model space Δ. There are two sources of ambiguity in return distribution: (1) parameter uncertainty; (2) latent state uncertainty. The first source declines over time as data accumulate, while the second does not. The investor learns the parameters and updates her belief over values of state variables over time, having in mind this structure of ambiguity. We calculate the posterior probability distribution of $N(\theta, \Sigma^{\mathbf{r}_t})$, and input the

[24] Among many studies, Bossaerts and Hillion (1999) compare a variety of stock return predictors and conclude that even the best prediction models have no out-of-sample forecasting power. Pesaran and Timmermann (1995) argue that predictability of stock returns is very low. Henriksson (1984) and Ferson and Schadt (1996) among others show that most mutual funds are not successful return timers. Following these studies, we assume constant expected return θ.

posterior statistics in the closed-form solution of optimal delegation given by Equation (14).

In the implementation, we assume $K = 1$. Investors' belief π_t is updated from August 1983 to September 2012 (350 months). The previous 685 months (July 1926 to July 1983) is used as training set to form the initial prior π_1 based on the smoothing algorithm (Gibbs sampler). The learning problem is solved by "particle filter", a recursive algorithm commonly used to estimate non-linear latent factor models. Due to its complexity, we provide the details on the estimation and calculation in a separate technical report that is available upon request.

Discussion: managers' knowledge. In the theoretical model, the fund managers know the probability distribution of returns. So, in the current setting, investors believe that the fund managers know exactly the true $N(\theta, \Sigma^{\mathbf{r}_t})$. In other words, at time t the fund manager's knowledge includes not only the time-invariant parameter values (θ, B, Ω and $\{(\alpha_k, \delta_k, \sigma_k^v)\}_{k=1}^K$) but also the true value of the state variable H_t.[25] The predictability of stock volatility has been shown by Andersen, Bollerslev, Christoffersen, and Diebold (2006) among others. Studies, such as Johannes, Korteweg, and Polson (2014) and Marquering and Verbeek (2004), demonstrate that volatility timing can add value to investors' portfolios. Busse (1999) shows that mutual fund managers time conditional market return volatility, and Chen and Liang (2007) show the same for hedge funds. Fund managers' ability to know the true parameter values and observe the volatilities is the extreme version of volatility timing. Investors' learning of H_t already exhibits a certain level of volatility timing, but investors assume that fund managers can do even better thanks to their better econometric models and access to broader sources of data.

4 Conclusion

Big data allows professional asset managers to better estimate the probability distribution of asset returns, but at the same time, demands specialization. It requires professionals to devote tremendous efforts to data collection and analysis. Therefore, big data also creates a division of knowledge – it has become increasingly difficult for investors to access data

[25]These are two extreme cases of knowledge. In the middle of the spectrum, for example, we may assume that investors understand the model structure but do not know the parameter values and state values, while fund managers know the model structure and parameter values but do not observe directly the state variable. The key is that the fund managers face less model uncertainty than the investors do.

sources as rich as professionals' or to understand their sophisticated analytical techniques. The starting point of our theoretical analysis is such informational difference between asset managers and investors. Specifically, managers know the true probability distribution of asset returns, while investors face model uncertainty, entertaining a set of candidate probability distributions.

Our framework does not feature frictions such as moral hazard. Managers dutifully allocate the delegated wealth on the efficient frontier. However, since investors do not know the true model, they have in mind that whichever model is true, managers form portfolio according to that model. As a result, the return on investors' delegated wealth becomes model-contingent. When allocating their retained wealth, investors hedge delegation uncertainty – they prefer (avoid) assets whose return distribution moves against (with) the efficient frontier across candidate models.

We solve the optimal delegation level and investors' portfolio choice by extending the quadratic representation of ambiguity preference by Maccheroni, Marinacci, and Ruffino (2013) into functional spaces. Fitting the estimated model uncertainty of investors into our solution, we find the model-implied delegation is reasonably correlated with its data counterpart, i.e., the fraction of wealth professionally managed in the U.S. stock market. We provide comparative statics, showing how the optimal level of delegation, which is also the size of asset management industry, varies with the number of risk factors, costs of asset management, investors' model uncertainty, and their preference parameters.

Investors' hedging against delegation uncertainty generates asset pricing implications that are distinct from the existing literature. The equilibrium average returns of assets deviate from CAPM by an ambiguity premium ("alpha"). Delegation fundamentally changes the nature of ambiguity. The hedging motive arises whether investors are ambiguity-averse or not, so alpha does not disappear even when investors are ambiguity-neutral. Moreover, we show that as the number of risk factors increases and the costs of asset management decline, the optimal level of delegation can approach 100%, but the equilibrium does not converge to CAPM. The more investors delegate, the stronger they hedge against delegation uncertainty. We provide supporting evidence. When delegation is unavailable, our model generates results that nest key findings in the literature of asset pricing models with ambiguity.

Our model reconciles the growth of asset management sector and the survival of anomaly strategies' alpha. It also provides practical guidance on finding investment strategies that deliver alpha in spite of increasing arbitrage capital. Moreover, we characterize the conditions under which our model generates delegation even if managers underperform the market index or deliver negative alpha to investors. This helps reconcile the mediocre performances of funds (at least on average) and the growth of asset management industry.

Table 1: Summary Statistics of Factor Returns and Institutional Ownership

This table shows the mean, median, standard deviation, count, quintile values and autocorrleation coefficient (ρ) of monthly returns and quarterly (relative) institutional ownership for each factor. The construction of the longshort factors returns and institutional ownership follows the Fama and French (1993) procedure for constructing HML and is described in detail in the text. Panel A summarizes monthly annualized factor returns. Panel B summarizes quarterly factor institutional ownership ($INST$) in percentage. The returns data is avaiable for 198001:201703 and the owernship data is available for 1980Q1:2016Q4.

	ACR	HML	BAB	CMA	IVOL	LTR	MOM	NI	RMW	STR	TVOL
Panel A: monthly return (annualized)											
count	447	447	447	447	447	447	447	447	447	447	447
mean	0.01	0.03	0.01	0.03	0.04	0.03	0.07	0.04	0.03	0.05	0.04
std	0.18	0.36	0.59	0.24	0.53	0.30	0.54	0.31	0.32	0.40	0.59
25%	-0.10	-0.19	-0.29	-0.12	-0.23	-0.16	-0.13	-0.11	-0.09	-0.14	-0.27
50%	0.01	0.01	0.01	0.02	0.02	0.01	0.07	0.02	0.03	0.02	0.03
75%	0.11	0.22	0.37	0.17	0.30	0.19	0.34	0.17	0.17	0.23	0.33
ρ	0.21	0.15	0.04	0.13	0.12	0.18	0.07	0.14	0.10	-0.03	0.08
Panel B: quarterly institutional ownership $INST$ (%)											
count	149	149	149	149	149	149	149	149	149	149	149
mean	-0.31	-0.76	-2.63	-1.02	-1.42	-1.03	0.51	-0.79	-0.92	-0.44	-1.70
std	0.71	1.00	1.49	0.89	1.50	1.69	1.43	1.31	0.87	1.39	1.29
25%	-0.74	-1.29	-3.29	-1.52	-2.29	-2.18	-0.28	-1.69	-1.49	-1.33	-2.54
50%	-0.36	-0.70	-2.28	-0.85	-1.47	-0.57	0.51	-0.71	-0.83	-0.46	-1.68
75%	-0.07	0.05	-1.57	-0.40	-0.76	0.10	1.41	-0.13	-0.42	0.52	-0.90
ρ	0.56	0.59	0.82	0.59	0.70	0.84	0.65	0.71	0.72	-0.04	0.55

Table 2: Predicting Future Factor Returns with Fund Ownership $INST$

This table shows predictive regressions of monthly long–short factor returns on lagged values of the factor (relative) institutional ownership ($INST$) controlling for other factor return predictors such as realized volatility RV. Panel A reports estimations from pooled OLS and fixed effect panel regressions:

$$R^{3m}_{i,t+1} = \alpha + \beta \cdot INST_{i,t} + \gamma \cdot X_{i,t} + \varepsilon_{i,t+1}$$

The left hand variable is monthly overlapping 3-month returns. Since ownership data is refreshed quarterly, standard errors are double-clustered at quarter and factor levels. Panel B reports estimations using Hodrick reverse predictive regressions

$$3 \times R^{1m}_{i,t+1} = \alpha + \beta \left(\frac{1}{3} \sum_{j=0}^{2} INST^{n}_{i,t-j} \right) + \gamma \cdot X_{i,t} + \varepsilon^{n}_{i,t+1}$$

The left hand variable is monthly non-overlapping returns multiplied by a factor of 3 to be compared with estimates from Panel A. The sample period is 198003:201612. Standard errors are in parentheses. *,**, and *** indicate 10%, 5% and 1% statistical significance respectively.

Panel A: panel regressions

	$R^{3m}_{i,t+1}$					
	(1)	(2)	(3)	(4)	(5)	(6)
$INST$	0.27***	0.21**	0.27***	0.31***	0.22*	0.28**
	(0.10)	(0.11)	(0.10)	(0.09)	(0.10)	(0.11)
RV				0.24	0.02	0.28
				(0.22)	(0.19)	(0.30)
Constant	0.01***			0.00		
	(0.00)			(0.01)		
Quarter FE		✓			✓	
Factor FE			✓			✓
Observations	4,884	4,884	4,884	4,513	4,513	4,513
Adjusted R^2	0.00	0.22	0.00	0.01	0.22	0.01
Residual Std. Error	0.06	0.06	0.06	0.06	0.06	0.06

Panel B: Hodrick (1992) reverse predictive regressions

	$3 \times R^{1m}_{i,t+1}$					
	(1)	(2)	(3)	(4)	(5)	(6)
$\frac{1}{3}\sum_{j=0}^{2} INST^{n}_{t-j}$	0.31***	0.28**	0.32**	0.36***	0.29***	0.34***
	(0.11)	(0.11)	(0.13)	(0.11)	(0.11)	(0.13)
RV				0.26***	0.03	0.30***
				(0.08)	(0.11)	(0.10)
Constant	0.01***			0.00		
	(0.00)			(0.00)		
Quarter FE		✓			✓	
Factor FE			✓			✓
Observations	4,884	4,884	4,884	4,535	4,535	4,535
Adjusted R^2	0.00	0.06	0.00	0.00	0.06	0.00
Residual Std. Error	0.10	0.10	0.10	0.11	0.10	0.11

Table 3: Summary Statistics: Equal-weighted Portfolios of Factors by Fund Ownership

This table shows the annualized mean, volatility, Sharpe ratio, skewness, kurtosis, best/worst month of the returns of portfolios of factors sorted on institutional owernship $INST$. At the end of each quarter, we rank all factors based on their institutional ownership $INST$. We long the top 4 factors ("H") and short the bottom 4 factors ("L") for the following 3 months, weighting each factor equally. We form an "M" portfolio by weighting the remaining medium $INST$ factors equally. The portfolio is rebalanced quarterly. The sample period is 198004:201703.

	H	M	L	H-L
Mean (ann.)	4.98%	2.68%	2.06%	2.91%
Vol (ann.)	6.16%	7.34%	11.38%	10.93%
Sharpe	0.81	0.36	0.18	0.27
Skewness	-0.18	-0.71	-0.42	0.96
Kurtosis	13.03	10.12	4.85	7.56
Observations	444	444	444	444

Figure 1: Cumulative Returns of Factors Sorted by Fund Ownership (Equal-Weighted)

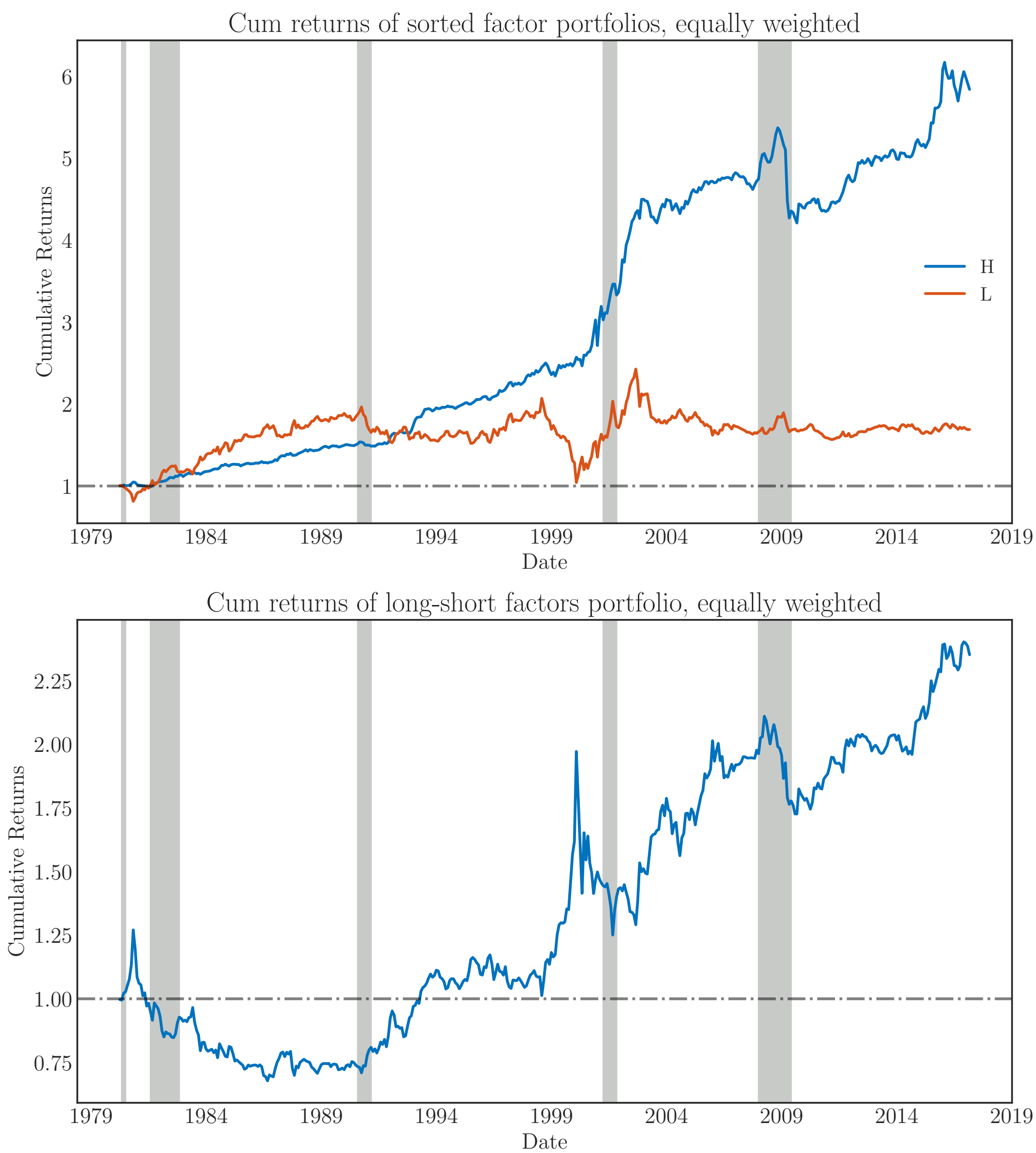


Figure 2: 60-month Rolling Average of Factor Returns Sorted by Fund Ownership, (Equal-Weighted)

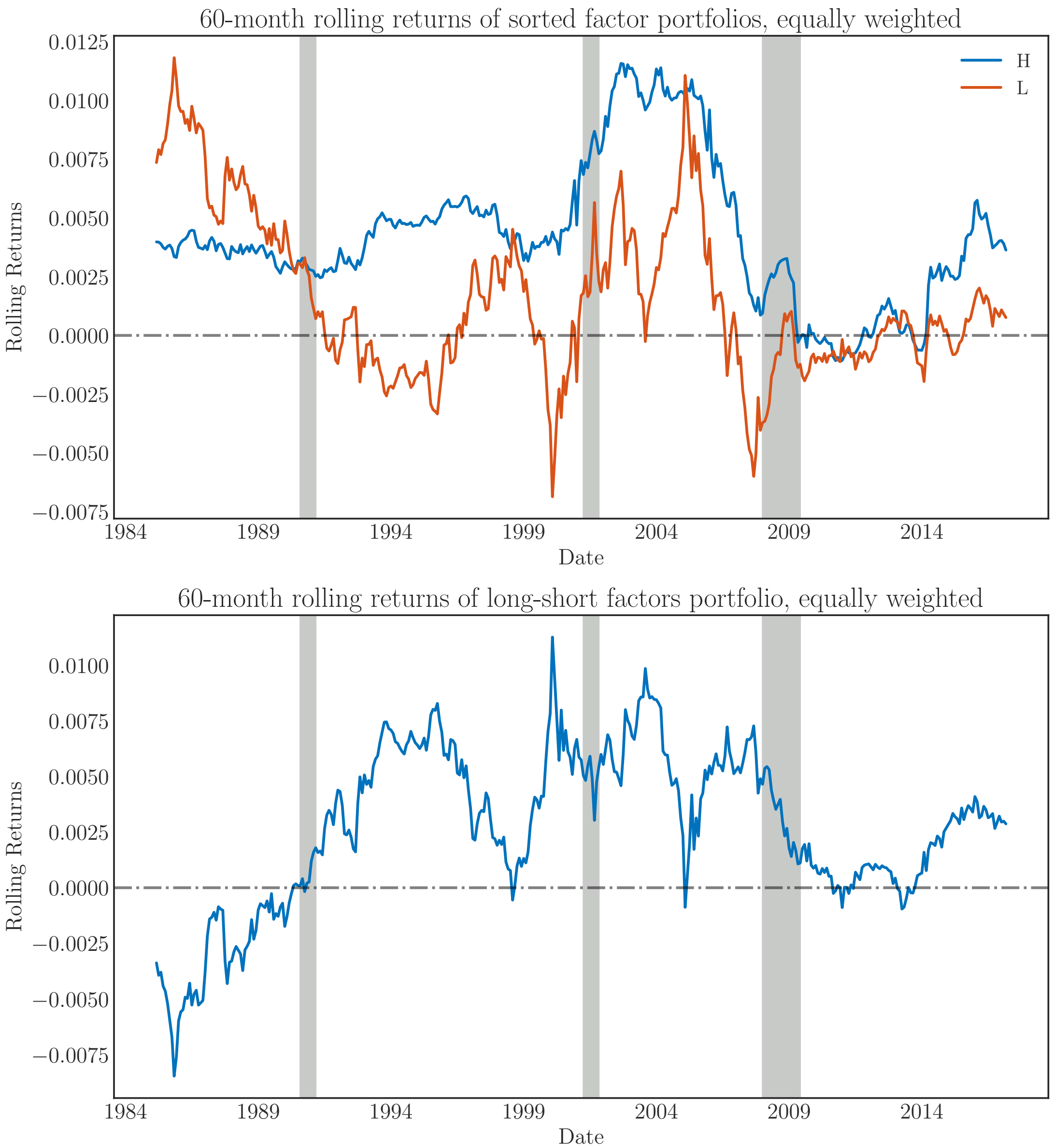


Figure 3: 60-month Rolling CAPM Alpha of Factors with High Delegation Ownership (Equal-weighted) and Aggregate Fund Ownership

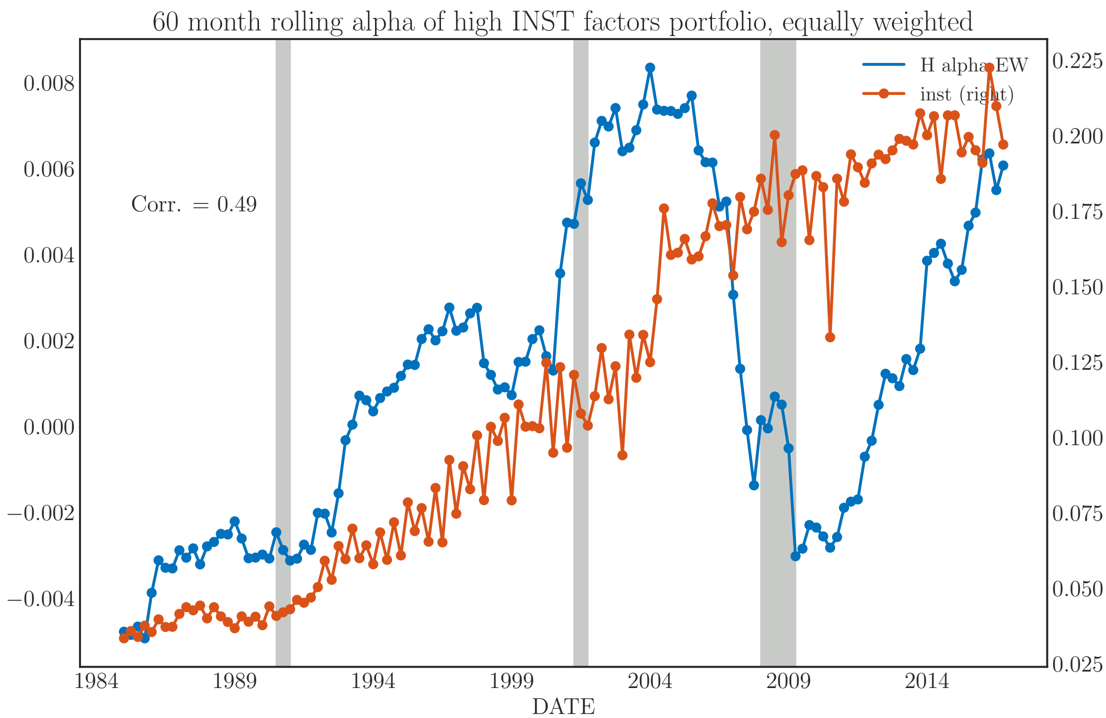


Figure 4: 60-month Rolling CAPM Alpha of Factor Long-Short portfolio (Equal-weighted) and Aggregate Delegation Ownership

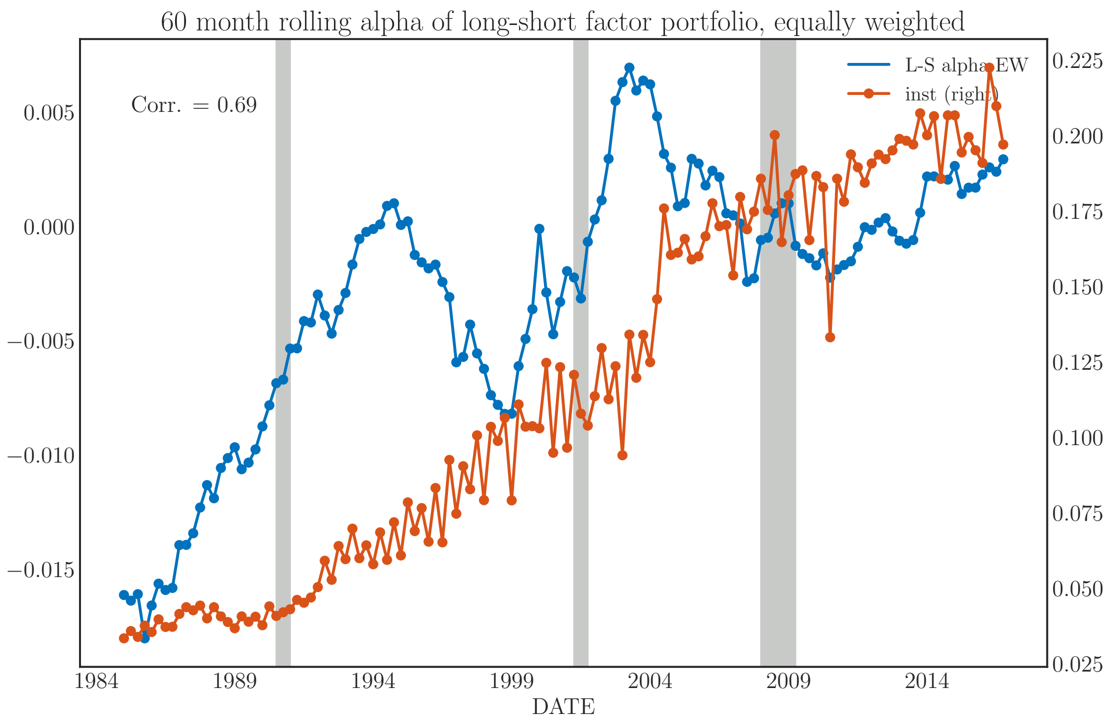
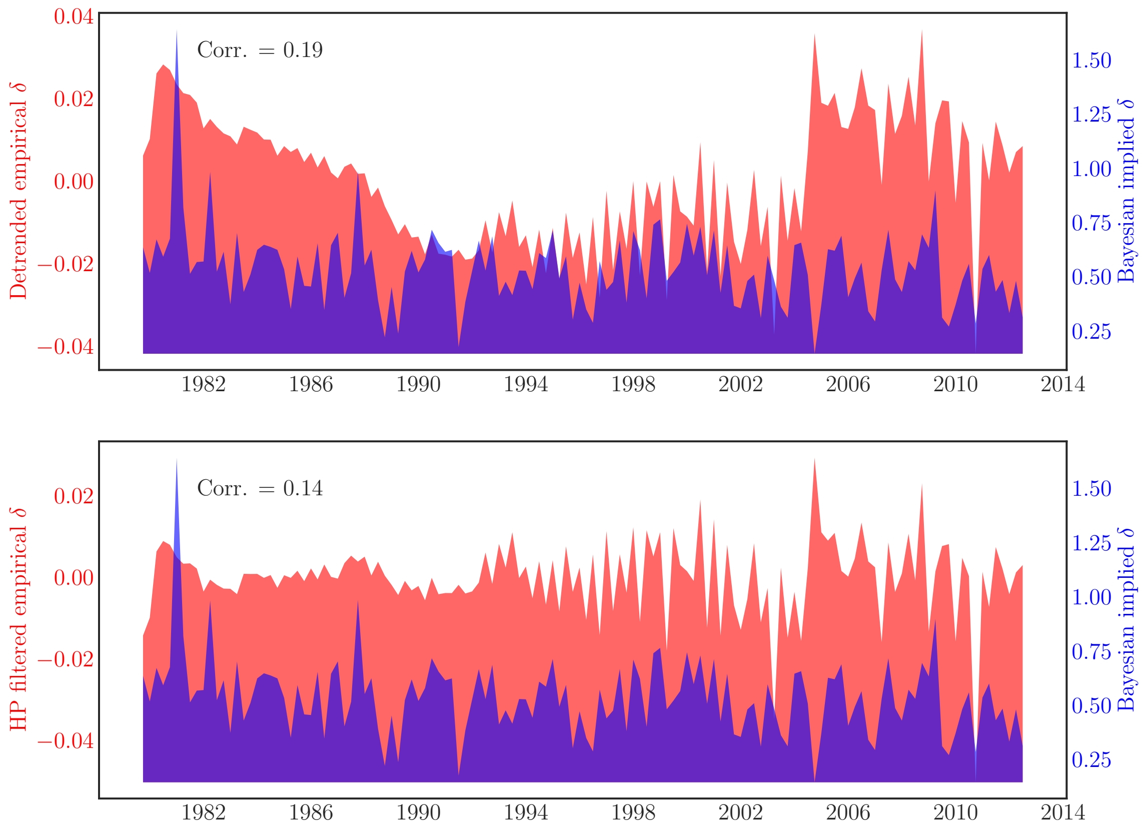


Figure 5: Model-implied and Detrended Empirical Fund Ownership δ



Appendix I: Quadratic Approximation

In the following, we use Q and q interchangeably to denote a candidate probability model. Define the following function corresponding to the certainty equivalent:

$$F\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right) = C\left(r_f + (\mathbf{r} - r_f \mathbf{1})^T \left[(1-\delta)\,\mathbf{w}^o + \delta \mathbf{w}^d(q)\right]\right)$$

Hence, $F : B(\mathbf{\Omega}, \mathbf{R}) \times \mathbf{R}^N \times L^\infty \mapsto \mathbf{R}$ is a functional defined on three Banach spaces, where $B(\mathbf{\Omega}, \mathbf{R})$ denotes the set of mappings from $\mathbf{\Omega}$ to $\mathbf{R}$.

Frechet derivatives of C. Here we list several useful expressions and definitions

- $\left(\upsilon^{-1}(\cdot)\right)' = \frac{1}{\upsilon'(\upsilon^{-1}(\cdot))}$ and $\phi'(\cdot) = \left(\upsilon \circ u^{-1}(\cdot)\right)' = \frac{\upsilon'\left(u^{-1}(\cdot)\right)}{u'(u^{-1}(\cdot))}$.
- $\left(\upsilon^{-1}(\cdot)\right)'' = -\frac{1}{[\upsilon'(\upsilon^{-1}(\cdot))]^2} \frac{\upsilon''\left(\upsilon^{-1}(\cdot)\right)}{\upsilon'(\upsilon^{-1}(\cdot))}$.
- $\phi''(\cdot) = \left(\upsilon \circ u^{-1}(\cdot)\right)'' = \frac{\upsilon'\left(u^{-1}(\cdot)\right)}{[u'(u^{-1}(\cdot))]^2} \left[\frac{\upsilon''\left(u^{-1}(\cdot)\right)}{\upsilon'(u^{-1}(\cdot))} - \frac{u''\left(u^{-1}(\cdot)\right)}{u'(u^{-1}(\cdot))}\right]$.
- Define $\gamma = -\frac{u''\left(r_f\right)}{u'\left(r_f\right)}$ and $\theta = -u'(r_f) \frac{\phi''\left(u\left(r_f\right)\right)}{\phi'\left(u\left(r_f\right)\right)} = -\left[\frac{\upsilon''\left(r_f\right)}{\upsilon'\left(r_f\right)} - \frac{u''\left(r_f\right)}{u'\left(r_f\right)}\right]$.
- Denote $D_{\mathbf{w}^o} F\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)$ and $D_{\mathbf{w}^d} F\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)$ to be the first order Frechet derivatives of C with respect to $\mathbf{w}^o$ and $\mathbf{w}^d$, and $D^2_{\mathbf{w}^o} F\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)$ and $D^2_{\mathbf{w}^d} F\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)$ to be the second order Frechet derivatives of C with respect to $\mathbf{w}^o$ and $\mathbf{w}^d$.
- Denote $V\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right) = \int_\Delta \phi\left(\int_\Omega u\left(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}\right) dQ(\omega)\right) d\pi(q)$, so $V(\mathbf{r}, \mathbf{0}, \mathbf{0}) = \phi(u(r_f))$.
- Denote $U\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q)\right) = \int_\Omega u\left(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}\right) dQ(\omega)$, so $U(\mathbf{r}, \mathbf{0}, \mathbf{0}) = u(r_f)$.
- For any random variable R and probability measure P, μ_P^R denotes the mean of R under P, Σ_P^R the covariance of R under P if R is vector and $\sigma_P^2(R)$ the variance under P if R is scalor.

Derivatives w.r.t. $\mathbf{w}^d$. First, calculate the Frechet derivatives of $V\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)$

$$\begin{aligned}
& D_{\mathbf{w}^d} V\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)(\boldsymbol{\delta}) \\
= & \int_\Delta \phi'\left(U\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q)\right)\right) \frac{\partial U\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q)\right)}{\partial \mathbf{w}^d(q)} \boldsymbol{\delta}(q)\, d\pi(q) \\
= & \int_\Delta \phi'\left(U\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q)\right)\right) \int_\Omega u'\left(r_{\delta, \mathbf{w}^o, \mathbf{w}^d}\right) \delta\left(\mathbf{r} - r_f \mathbf{1}\right)^T \boldsymbol{\delta}(q)\, dQ(\omega)\, d\pi(q)
\end{aligned}$$

which is a row vector, and

$$\begin{aligned}
& D^2_{\mathbf{w}^d} V\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) \\
= \; & \int_\Delta \phi''\left(U\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q)\right)\right)\left(\int_\Omega u'\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right)\delta\left(\mathbf{r}-r_f\mathbf{1}\right)^T \boldsymbol{\delta}_2(q)\, dQ(\omega)\right) \\
& \left(\int_\Omega u'\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right)\delta\left(\mathbf{r}-r_f\mathbf{1}\right)^T \boldsymbol{\delta}_1(q)\, dQ(\omega)\right) d\pi(q) + \int_\Delta \phi'\left(U\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q)\right)\right) \\
& \int_\Omega u''\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right)\delta^2 \boldsymbol{\delta}_1(q)^T\left(\mathbf{r}-r_f\mathbf{1}\right)\left(\mathbf{r}-r_f\mathbf{1}\right)^T \boldsymbol{\delta}_2(q)\, dQ(\omega)\, d\pi(q)
\end{aligned}$$

which is a N-by-N matrix. Evaluate at $\left(\mathbf{w}^o, \mathbf{w}^d\right) = \mathbf{0}$ and $\boldsymbol{\delta} = \boldsymbol{\delta}_1 = \boldsymbol{\delta}_1 = \mathbf{w}^d$:

$$\begin{aligned}
D_{\mathbf{w}^d} V(\mathbf{r}, \mathbf{0}, \mathbf{0})\left(\mathbf{w}^d\right) &= \upsilon'(r_f)\,\delta E_\pi\left(E_Q\left(\left(\mathbf{r}-r_f\mathbf{1}\right)^T \mathbf{w}^d(q)\right)\right) \\
D^2_{\mathbf{w}^d} V(\mathbf{r}, \mathbf{0}, \mathbf{0}) &= \phi''(u(r_f))\left[u'(r_f)\right]^2 \delta^2 E_\pi\left(\left[E_Q\left(\left(\mathbf{r}-r_f\mathbf{1}\right)^T \mathbf{w}^d(q)\right)\right]^2\right) + \\
& \quad \phi'(u(r_f))\, u''(r_f)\left(\delta^2\right) E_\pi\left(E_Q\left(\left[\left(\mathbf{r}-r_f\mathbf{1}\right)^T \mathbf{w}^d(q)\right]^2\right)\right)
\end{aligned}$$

By chain rule,

$$\begin{aligned}
& D_{\mathbf{w}^d} F\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)(\boldsymbol{\delta}) = \frac{D_{\mathbf{w}^d} V\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)(\boldsymbol{\delta})}{\upsilon'\left(\upsilon^{-1}\left(V\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)\right)\right)} \\
= \; & \int_\Delta \frac{\phi'\left(U\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d(q)\right)\right)}{\upsilon'\left(\upsilon^{-1}\left(V\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)\right)\right)} \int_\Omega u'\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right)\delta\left(\mathbf{r}-r_f\mathbf{1}\right)^T \boldsymbol{\delta}(q)\, dQ(\omega)\, d\pi(q) \\
& D^2_{\mathbf{w}^d} F\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) = -\frac{\upsilon''\left(\upsilon^{-1}\left(V\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)\right)\right)}{\left[\upsilon'\left(\upsilon^{-1}\left(V\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)\right)\right)\right]^3}\left[D_{\mathbf{w}^d} V\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)(\boldsymbol{\delta}_1)\right] \\
& \left[D_{\mathbf{w}^d} V\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)(\boldsymbol{\delta}_2)\right] + \frac{D^2_{\mathbf{w}^d} V\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{\upsilon'\left(\upsilon^{-1}\left(V\left(\mathbf{r}, \mathbf{w}^o, \mathbf{w}^d\right)\right)\right)}
\end{aligned}$$

Evaluate at $\left(\mathbf{w}^o, \mathbf{w}^d\right) = \mathbf{0}$ and $\boldsymbol{\delta} = \boldsymbol{\delta}_1 = \boldsymbol{\delta}_1 = \mathbf{w}^d$:

$$\begin{aligned}
D_{\mathbf{w}^d} F(\mathbf{r}, \mathbf{0}, \mathbf{0})\left(\mathbf{w}^d\right) &= \delta E_\pi\left(E_Q\left(\left(\mathbf{r}-r_f\mathbf{1}\right)^T \mathbf{w}^d(q)\right)\right) \\
D^2_{\mathbf{w}^d} F(\mathbf{r}, \mathbf{0}, \mathbf{0})\left(\mathbf{w}^d, \mathbf{w}^d\right) &= -\theta\delta^2 Var_\pi\left(E_Q\left(\left(\mathbf{r}-r_f\mathbf{1}\right)^T \mathbf{w}^d(q)\right)\right) - \\
& \quad \gamma\delta^2 E_\pi\left(\sigma^2_Q\left(\left(\mathbf{r}-r_f\mathbf{1}\right)^T \mathbf{w}^d(q)\right)\right)
\end{aligned}$$

Derivatives w.r.t. $\mathbf{w}^o$. First, calculate the Frechet derivatives of $V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)$:

$$\begin{aligned}
& D_{\mathbf{w}^o}V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)(\boldsymbol{\delta}) \\
= & \int_\Delta \phi'\left(U\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d(q)\right)\right)\frac{\partial U\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d(q)\right)}{\partial \mathbf{w}^o}\boldsymbol{\delta}d\pi(q) \\
= & \int_\Delta \phi'\left(U\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d(q)\right)\right)\int_\Omega u'\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right)(1-\delta)\left(\mathbf{r}-r_f\mathbf{1}\right)^T\boldsymbol{\delta}dQ(\omega)\,d\pi(q)
\end{aligned}$$

which is a row vector, and

$$\begin{aligned}
& D^2_{\mathbf{w}^o}V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)(\boldsymbol{\delta}_1,\boldsymbol{\delta}_2) \\
= & \int_\Delta \phi''\left(U\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d(q)\right)\right)\left(\int_\Omega u'\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right)(1-\delta)\left(\mathbf{r}-r_f\mathbf{1}\right)^T\boldsymbol{\delta}_1dQ(\omega)\right) \\
& \left(\int_\Omega u'\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right)(1-\delta)\left(\mathbf{r}-r_f\mathbf{1}\right)^T\boldsymbol{\delta}_2dQ(\omega)\right)d\pi(q)+ \\
& \int_\Delta \phi'\left(U\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d(q)\right)\right)\int_\Omega u''\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right)(1-\delta)^2 \\
& \boldsymbol{\delta}_1^T\left(\mathbf{r}-r_f\mathbf{1}\right)\left(\mathbf{r}-r_f\mathbf{1}\right)^T\boldsymbol{\delta}_2dQ(\omega)\,d\pi(q)
\end{aligned}$$

which is a N-by-N matrix. Evaluate at $\left(\mathbf{w}^o,\mathbf{w}^d\right)=\mathbf{0}$ and $\boldsymbol{\delta}=\boldsymbol{\delta}_1=\boldsymbol{\delta}_2=\mathbf{w}^o$:

$$\begin{aligned}
D_{\mathbf{w}^o}V(\mathbf{r},\mathbf{0},\mathbf{0})(\mathbf{w}^o) &= (1-\delta)\,\upsilon'(r_f)\,E_{\overline{Q}}\left(\left(\mathbf{r}-r_f\mathbf{1}\right)^T\mathbf{w}^o\right) \\
D^2_{\mathbf{w}^o}V(\mathbf{r},\mathbf{0},\mathbf{0})(\mathbf{w}^o,\mathbf{w}^o) &= \phi''(u(r_f))\left[u'(r_f)\right]^2(1-\delta)^2E_\pi\left(\left[E_Q\left(\left(\mathbf{r}-r_f\mathbf{1}\right)^T\mathbf{w}^o\right)\right]^2\right) \\
&\quad +\phi'(u(r_f))\,u''(r_f)(1-\delta)^2E_{\overline{Q}}\left(\left[\left(\mathbf{r}-r_f\mathbf{1}\right)^T\mathbf{w}^o\right]^2\right)
\end{aligned}$$

Then,

$$\begin{aligned}
D_{\mathbf{w}^o}F\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)(\boldsymbol{\delta}) &= \frac{D_{\mathbf{w}^o}V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)(\boldsymbol{\delta})}{\upsilon'\left(\upsilon^{-1}\left(V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)\right)\right)} \\
&= \frac{1}{\upsilon'\left(\upsilon^{-1}\left(V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)\right)\right)}\int_\Delta \phi'\left(U\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d(q)\right)\right) \\
&\quad \int_\Omega u'\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right)(1-\delta)\left(\mathbf{r}-r_f\mathbf{1}\right)^T\boldsymbol{\delta}dQ(\omega)\,d\pi(q)
\end{aligned}$$

and

$$\begin{aligned}
& D^2_{\mathbf{w}^o} F\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)(\boldsymbol{\delta}_1,\boldsymbol{\delta}_2) \\
= & -\frac{\upsilon''\left(\upsilon^{-1}\left(V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)\right)\right)}{\left[\upsilon'\left(\upsilon^{-1}\left(V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)\right)\right)\right]^3}\left[D_{\mathbf{w}^o} V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)(\boldsymbol{\delta}_1)\right]\left[D_{\mathbf{w}^o} V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)(\boldsymbol{\delta}_2)\right] \\
& +\frac{D^2_{\mathbf{w}^o} V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)(\mathbf{w}^o,\mathbf{w}^o)}{\upsilon'\left(\upsilon^{-1}\left(V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)\right)\right)}
\end{aligned}$$

Evaluate at $\left(\mathbf{w}^o,\mathbf{w}^d\right)=\mathbf{0}$ and $\boldsymbol{\delta}=\boldsymbol{\delta}_1=\boldsymbol{\delta}_2=\mathbf{w}^o$:

$$\begin{aligned}
D_{\mathbf{w}^o} F(\mathbf{r},\mathbf{0},\mathbf{0})(\mathbf{w}^o) &= (1-\delta)\left(\mu^{\mathbf{r}}_{\overline{Q}}-r_f\mathbf{1}\right)^T\mathbf{w}^o \\
D^2_{\mathbf{w}^o} F(\mathbf{r},\mathbf{0},\mathbf{0})(\mathbf{w}^o,\mathbf{w}^o) &= -\theta(1-\delta)^2 Var_\pi\left(E_Q\left((\mathbf{r}-r_f\mathbf{1})^T\mathbf{w}^o\right)\right)- \\
& \quad \gamma(1-\delta)^2 Var_{\overline{Q}}\left((\mathbf{r}-r_f\mathbf{1})^T\mathbf{w}^o\right)
\end{aligned}$$

Second Derivatives w.r.t. $\mathbf{w}^d$ and $\mathbf{w}^o$. Finally,

$$\begin{aligned}
& D^2_{\mathbf{w}^o\mathbf{w}^d} F\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)(\boldsymbol{\delta}_1,\boldsymbol{\delta}_2) \\
= & \frac{D_{\mathbf{w}^o\mathbf{w}^d} V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)(\boldsymbol{\delta}_1,\boldsymbol{\delta}_2)}{\upsilon'\left(\upsilon^{-1}\left(V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)\right)\right)}-\frac{\left[\upsilon''\left(\upsilon^{-1}\left(V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)\right)\right)/\upsilon'\left(\upsilon^{-1}\left(V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)\right)\right)\right]}{\left[\upsilon'\left(\upsilon^{-1}\left(V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)\right)\right)\right]^2} \\
& \left[D_{\mathbf{w}^o} V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)(\boldsymbol{\delta}_1)\right]\left[D_{\mathbf{w}^d} V\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right)(\boldsymbol{\delta}_2)\right]
\end{aligned}$$

Evaluate at $\left(\mathbf{w}^o,\mathbf{w}^d\right)=\mathbf{0}$ and $\boldsymbol{\delta}_1=\mathbf{w}^o$, $\boldsymbol{\delta}_2=\mathbf{w}^d$:

$$\begin{aligned}
& D^2_{\mathbf{w}^o\mathbf{w}^d} F(\mathbf{r},\mathbf{0},\mathbf{0})\left(\mathbf{w}^o,\mathbf{w}^d\right) \\
= & \frac{D_{\mathbf{w}^o\mathbf{w}^d} V(\mathbf{r},\mathbf{0},\mathbf{0})\left(\mathbf{w}^o,\mathbf{w}^d\right)}{\upsilon'(r_f)}-\frac{\upsilon''(r_f)}{\left[\upsilon'(r_f)\right]^3}\left[D_{\mathbf{w}^o} V(\mathbf{r},\mathbf{0},\mathbf{0})(\mathbf{w}^o)\right]\left[D_{\mathbf{w}^d} V(\mathbf{r},\mathbf{0},\mathbf{0})\left(\mathbf{w}^d\right)\right]
\end{aligned}$$

where

$$\begin{aligned}
& D_{\mathbf{w}^o\mathbf{w}^d} V(\mathbf{r},\mathbf{0},\mathbf{0})\left(\mathbf{w}^o,\mathbf{w}^d\right) \\
= & -\upsilon'(r_f)\,\theta(1-\delta)\,\delta\int_\Delta \mathbf{w}^d(q)^T E_Q(\mathbf{r}-r_f\mathbf{1})\,E_Q\left((\mathbf{r}-r_f\mathbf{1})^T\right)\mathbf{w}^o d\pi(q) \\
& -\upsilon'(r_f)\,\gamma(1-\delta)\,\delta\int_\Delta \mathbf{w}^{oT} E_Q\left((\mathbf{r}-r_f\mathbf{1})(\mathbf{r}-r_f\mathbf{1})^T\right)\mathbf{w}^d(q)\,d\pi(q)
\end{aligned}$$

Simplify the expression:

$$\begin{aligned}
& D^2_{\mathbf{w}^o\mathbf{w}^d} F(\mathbf{r},\mathbf{0},\mathbf{0})\left(\mathbf{w}^o,\mathbf{w}^d\right) \\
= & \frac{D_{\mathbf{w}^o\mathbf{w}^d} V(\mathbf{r},\mathbf{0},\mathbf{0})\left(\mathbf{w}^o,\mathbf{w}^d\right)}{\upsilon'(r_f)} - \frac{\upsilon''(r_f)}{[\upsilon'(r_f)]^3}\left[D_{\mathbf{w}^o} V(\mathbf{r},\mathbf{0},\mathbf{0})(\mathbf{w}^o)\right]\left[D_{\mathbf{w}^d} V(\mathbf{r},\mathbf{0},\mathbf{0})\left(\mathbf{w}^d\right)\right] \\
= & -(\theta+\gamma)(1-\delta)\delta cov_\pi\left(E_Q\left((\mathbf{r}-r_f\mathbf{1})^T\mathbf{w}^o\right), E_Q\left((\mathbf{r}-r_f\mathbf{1})^T\mathbf{w}^d(q)\right)\right) \\
& -\gamma(1-\delta)\delta E_\pi\left(cov_Q\left((\mathbf{r}-r_f\mathbf{1})^T\mathbf{w}^o, (\mathbf{r}-r_f\mathbf{1})^T\mathbf{w}^d(q)\right)\right)
\end{aligned}$$

Taylor expansion of C. By Theorem 8.16 of Jost (2005),

$$\begin{aligned}
& C\left(r_f + (\mathbf{r}-r_f\mathbf{1})^T\left[(1-\delta)\mathbf{w}^o + \delta\mathbf{w}^d(q)\right]\right) = F\left(\mathbf{r},\mathbf{w}^o,\mathbf{w}^d\right) \\
& = r_f + D_{\mathbf{w}^o} F(\mathbf{r},\mathbf{0},\mathbf{0})(\mathbf{w}^o) + D_{\mathbf{w}^d} F(\mathbf{r},\mathbf{0},\mathbf{0})\left(\mathbf{w}^d\right) \\
& + \frac{1}{2} D^2_{\mathbf{w}^o} F(\mathbf{r},\mathbf{0},\mathbf{0})(\mathbf{w}^o,\mathbf{w}^o) + \frac{1}{2} D^2_{\mathbf{w}^d} F(\mathbf{r},\mathbf{0},\mathbf{0})\left(\mathbf{w}^d,\mathbf{w}^d\right) \\
& + D^2_{\mathbf{w}^o\mathbf{w}^d} F(\mathbf{r},\mathbf{0},\mathbf{0})\left(\mathbf{w}^o,\mathbf{w}^d\right) + R\left(\mathbf{w}^o,\mathbf{w}^d\right)
\end{aligned}$$

where $D_{\mathbf{w}^d} F(\mathbf{r},\mathbf{0},\mathbf{0})$, $D^2_{\mathbf{w}^d} F(\mathbf{r},\mathbf{0},\mathbf{0})$, $D_{\mathbf{w}^o} F(\mathbf{r},\mathbf{0},\mathbf{0})$, $D^2_{\mathbf{w}^o} F(\mathbf{r},\mathbf{0},\mathbf{0})$, and $D^2_{\mathbf{w}^o\mathbf{w}^d} F(\mathbf{r},\mathbf{0},\mathbf{0})$ have been solved and $\lim_{\left(\mathbf{w}^o,\mathbf{w}^d\right)\to\mathbf{0}} \frac{R\left(\mathbf{w}^o,\mathbf{w}^d\right)}{\left\|\left(\mathbf{w}^o,\mathbf{w}^d\right)\right\|^2} = 0$.

To simplify the notations, let $R^{\mathbf{w}}$ denote the *excess* return generated by any portfolio $\mathbf{w}$ and let $R^{\mathbf{w}}_P$ denotes the expected *excess* return of any portfolio $\mathbf{w}$ under probability measure P. Also notice that $\mathbf{w}^d(q) = \left(\gamma\Sigma^{\mathbf{r}}_Q\right)^{-1}\left(\mu^{\mathbf{r}}_Q - r_f\mathbf{1}\right)$. We have:

$$E_\pi\left(cov_Q\left(R^{\mathbf{w}^o}, R^{\mathbf{w}^d}\right)\right) = E_\pi\left(\mathbf{w}^{oT}\Sigma^{\mathbf{r}}_Q\mathbf{w}^d(q)\right) = \frac{1}{\gamma}\left(\mu^{\mathbf{r}}_Q - r_f\mathbf{1}\right)^T\mathbf{w}^o = \frac{1}{\gamma}R^{\mathbf{w}^o}_Q$$

Taylor expansion can be simplified as

$$\begin{aligned}
& C\left(r_f + (\mathbf{r}-r_f\mathbf{1})^T\left[(1-\delta)\mathbf{w}^o + \delta\mathbf{w}^d(q)\right]\right) \\
\approx & \; r_f + (1-\delta)^2 R^{\mathbf{w}^o}_Q + \delta E_\pi\left(R^{\mathbf{w}^d}_Q\right) - (\theta+\gamma)(1-\delta)\delta cov_\pi\left(R^{\mathbf{w}^o}_Q, R^{\mathbf{w}^d}_Q\right) \\
& -\frac{(1-\delta)^2}{2}\left(\gamma\sigma^2_Q\left(R^{\mathbf{w}^o}\right) + \theta\sigma^2_\pi\left(R^{\mathbf{w}^o}_Q\right)\right) - \frac{\delta^2}{2}\left(\gamma E_\pi\left(\sigma^2_Q\left(R^{\mathbf{w}^d}\right)\right) + \theta\sigma^2_\pi\left(R^{\mathbf{w}^d}_Q\right)\right)
\end{aligned}$$

Appendix II: Optimal Portfolio and Delegation

The investor's problem is

$$\max_{\mathbf{w}^o,\delta} C\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right) - \delta\psi$$

given that

$$\mathbf{w}^d(q) = \left(\gamma\Sigma^{\mathbf{r}}_Q\right)^{-1}\left(\mu^{\mathbf{r}}_Q - r_f\mathbf{1}\right)$$

Approximate $C\left(r_{\delta,\mathbf{w}^o,\mathbf{w}^d}\right)$:

$$
\begin{aligned}
& C\left(r_f + (\mathbf{r} - r_f\mathbf{1})^T\left[(1-\delta)\,\mathbf{w}^o + \delta\mathbf{w}^d\right]\right) \\
\approx\ & r_f + (1-\delta)^2\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)^T\mathbf{w}^o - (\theta+\gamma)(1-\delta)\,\delta cov_\pi\left(\mu_Q^{\mathbf{r}} - r_f\mathbf{1}, R_Q^{\mathbf{w}^d}\right)^T\mathbf{w}^o \\
& -\frac{(1-\delta)^2}{2}\left(\gamma\mathbf{w}^{oT}\Sigma_{\overline{Q}}^{\mathbf{r}}\mathbf{w}^o + \theta\mathbf{w}^{oT}\Sigma_\pi^{\mu_Q^{\mathbf{r}}}\mathbf{w}^o\right) + \delta E_\pi\left(R_Q^{\mathbf{w}^d}\right) \\
& -\frac{\delta^2}{2}\left(\gamma E_\pi\left(\sigma_Q^2\left(R^{\mathbf{w}^d}\right)\right) + \theta\sigma_\pi^2\left(R_Q^{\mathbf{w}^d}\right)\right)
\end{aligned}
$$

The first order condition of $\mathbf{w}^o$:

$$
\mathbf{w}^o = \left(\gamma\Sigma_{\overline{Q}}^{\mathbf{r}} + \theta\Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)^{-1}\left[\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right) - (\theta+\gamma)\frac{\delta}{1-\delta}cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)\right]
$$

From the first order condition of δ, δ equal to

$$
\frac{\gamma\sigma_{\overline{Q}}^2\left(R^{\mathbf{w}^o}\right) + \theta\sigma_\pi^2\left(R_Q^{\mathbf{w}^o}\right) + E_\pi\left(R_Q^{\mathbf{w}^d}\right) - 2R_{\overline{Q}}^{\mathbf{w}^o} - (\theta+\gamma)\,cov_\pi\left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d}\right) - \psi}{\gamma\sigma_{\overline{Q}}^2\left(R^{\mathbf{w}^o}\right) + \theta\sigma_\pi^2\left(R_Q^{\mathbf{w}^o}\right) + E_\pi\left(R_Q^{\mathbf{w}^d}\right) + \theta\sigma_\pi^2\left(R_Q^{\mathbf{w}^d}\right) - 2R_{\overline{Q}}^{\mathbf{w}^o} - 2(\theta+\gamma)\,cov_\pi\left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d}\right)},
$$

where $\gamma\sigma_{\overline{Q}}^2\left(R^{\mathbf{w}^o}\right) + \theta\sigma_\pi^2\left(R_Q^{\mathbf{w}^o}\right)$ can be simplified because

$$
\begin{aligned}
& \mathbf{w}^{oT}\left[\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right) - (\theta+\gamma)\frac{\delta}{1-\delta}cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)\right] \\
=\ & R_{\overline{Q}}^{\mathbf{w}^o} - (\theta+\gamma)\frac{\delta}{1-\delta}cov_\pi\left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d}\right)
\end{aligned}
$$

So,

$$
1-\delta = \frac{\theta\sigma_\pi^2\left(R_Q^{\mathbf{w}^d}\right) - (\theta+\gamma)\,cov_\pi\left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d}\right) + \psi}{E_\pi\left(R_Q^{\mathbf{w}^d}\right) + \theta\sigma_\pi^2\left(R_Q^{\mathbf{w}^d}\right) - R_{\overline{Q}}^{\mathbf{w}^o} - \left(\frac{2-\delta}{1-\delta}\right)(\theta+\gamma)\,cov_\pi\left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d}\right)}
$$

Divide both sides by $1-\delta$ and rearrange: δ is equal to

$$
\frac{E_\pi\left(R_Q^{\mathbf{w}^d}\right) - R_{\overline{Q}}^{\mathbf{w}^o} - (\theta+\gamma)\,cov_\pi\left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d}\right) - \psi}{E_\pi\left(R_Q^{\mathbf{w}^d}\right) + \theta\sigma_\pi^2\left(R_Q^{\mathbf{w}^d}\right) - R_{\overline{Q}}^{\mathbf{w}^o} - (\theta+\gamma)\,cov_\pi\left(R_Q^{\mathbf{w}^o}, R_Q^{\mathbf{w}^d}\right)}
$$

Appendix III: Analysis under the Simplified Ambiguity

Optimal delegation and portfolio. First, we rewrite $cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)$ under the three assumptions that simplify the structure of model uncertainty. The expected delegation return under probability model Q is

$$R_Q^{\mathbf{w}^d} = \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1}\right)^T \mathbf{w}^d (Q) = \frac{1}{\gamma} \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1}\right)^T \left(\Sigma_P^{\mathbf{r}}\right)^{-1} \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1}\right)$$

We can rewrite and decompose the cross-model covariance between the expected asset returns and the expected delegation return as follows.

$$\begin{aligned} cov_\pi \left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right) &= cov_\pi \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right) \\ &= \frac{1}{\gamma} cov_\pi \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}, \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)^T \left(\Sigma_P^{\mathbf{r}}\right)^{-1} \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)\right) \\ &\quad + \frac{1}{\gamma} cov_\pi \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}, \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)^T \left(\Sigma_P^{\mathbf{r}}\right)^{-1} \left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right)\right) \\ &\quad + \frac{1}{\gamma} cov_\pi \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}, \left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right)^T \left(\Sigma_P^{\mathbf{r}}\right)^{-1} \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1}\right)\right) \end{aligned}$$

To proceed, first, we recorgnize that $\left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)^T \left(\Sigma_P^{\mathbf{r}}\right)^{-1} \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)$ is a linear combination of $\left(\mu_Q^{\mathbf{r}_i} - \mu_{\overline{Q}}^{\mathbf{r}_i}\right)\left(\mu_Q^{\mathbf{r}_j} - \mu_{\overline{Q}}^{\mathbf{r}_j}\right)$ weighted by the elements of $\left(\Sigma_P^{\mathbf{r}}\right)^{-1}$. Under the assumption that π is Gaussian, we use Isserlis' theorem to eliminate the first term. For any asset k,

$$\begin{aligned} & cov_\pi \left(\mu_Q^{\mathbf{r}_k} - \mu_{\overline{Q}}^{\mathbf{r}_k}, \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)^T \left(\Sigma_P^{\mathbf{r}}\right)^{-1} \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)\right) \\ &= \sum_{i,j} \left(\Sigma_P^{\mathbf{r}}\right)^{-1}_{(i,j)} \left(E_\pi \left[\left(\mu_Q^{\mathbf{r}_k} - \mu_{\overline{Q}}^{\mathbf{r}_k}\right)\left(\mu_Q^{\mathbf{r}_i} - \mu_{\overline{Q}}^{\mathbf{r}_i}\right)\left(\mu_Q^{\mathbf{r}_j} - \mu_{\overline{Q}}^{\mathbf{r}_j}\right)\right]\right. \\ & \left. - E_\pi \left(\mu_Q^{\mathbf{r}_k} - \mu_{\overline{Q}}^{\mathbf{r}_k}\right) E \left[\left(\mu_Q^{\mathbf{r}_i} - \mu_{\overline{Q}}^{\mathbf{r}_i}\right)\left(\mu_Q^{\mathbf{r}_j} - \mu_{\overline{Q}}^{\mathbf{r}_j}\right)\right]\right) = 0, \end{aligned}$$

because first, $E_\pi \left[\left(\mu_Q^{\mathbf{r}_k} - \mu_{\overline{Q}}^{\mathbf{r}_k}\right)\left(\mu_Q^{\mathbf{r}_i} - \mu_{\overline{Q}}^{\mathbf{r}_i}\right)\left(\mu_Q^{\mathbf{r}_j} - \mu_{\overline{Q}}^{\mathbf{r}_j}\right)\right]$ is the expectation of three zero-mean normal random variables, and thus, is equal zero, and second, $E_\pi \left(\mu_Q^{\mathbf{r}_k} - \mu_{\overline{Q}}^{\mathbf{r}_k}\right) = E_\pi \left(\mu_Q^{\mathbf{r}_k}\right) - \mu_{\overline{Q}}^{\mathbf{r}_k} = \mu_{\overline{Q}}^{\mathbf{r}_k} - \mu_{\overline{Q}}^{\mathbf{r}_k} = 0$.

Therefore, we have

$$\begin{aligned} & cov_\pi \left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right) \\ = & \frac{1}{\gamma} cov_\pi \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}, \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)^T \left(\Sigma_P^{\mathbf{r}}\right)^{-1} \left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right)\right) \\ & + \frac{1}{\gamma} cov_\pi \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}, \left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f \mathbf{1}\right)^T \left(\Sigma_P^{\mathbf{r}}\right)^{-1} \left(\mu_Q^{\mathbf{r}} - r_f \mathbf{1}\right)\right). \end{aligned}$$

Using the fact that $(\Sigma_P^{\mathbf{r}})^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)$ is a constant vector, we can rewrite the first term as

$$\begin{aligned} & \frac{1}{\gamma} cov_\pi\left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}, \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)^T (\Sigma_P^{\mathbf{r}})^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)\right) \\ = & \frac{1}{\gamma} cov_\pi\left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}, \left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)\right)^T (\Sigma_P^{\mathbf{r}})^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right) \\ = & \frac{1}{\gamma}\left(\Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)(\Sigma_P^{\mathbf{r}})^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right). \end{aligned}$$

For the second term, we can replace $\left(\mu_Q^{\mathbf{r}} - r_f\mathbf{1}\right)$ with $\left(\mu_Q^{\mathbf{r}} - \mu_{\overline{Q}}^{\mathbf{r}}\right)$ because both $\mu_{\overline{Q}}^{\mathbf{r}}$ and $r_f\mathbf{1}$ are constant vectors, so this term is exactly the same as the first term. Therefore, we have

$$cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right) = \frac{2}{\gamma}\left(\Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)(\Sigma_P^{\mathbf{r}})^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)$$

Under the assumption that $\Sigma_\pi^{\mu_Q^{\mathbf{r}}} = \upsilon\Sigma_P^{\mathbf{r}}$,

$$cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right) = \frac{2}{\gamma}(\upsilon\Sigma_P^{\mathbf{r}})(\Sigma_P^{\mathbf{r}})^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right) = \frac{2\upsilon}{\gamma}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right),$$

and the investor's portfolio is

$$\begin{aligned} \mathbf{w}^o &= \left(\gamma\Sigma_{\overline{Q}}^{\mathbf{r}} + \theta\Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)^{-1}\left[\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right) - (\theta+\gamma)\left(\frac{\delta}{1-\delta}\right) cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)\right] \\ &= (\Sigma_P^{\mathbf{r}})^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)\left[\left(\frac{1}{\gamma+\upsilon\theta}\right) - \left(\frac{\gamma+\theta}{\gamma+\upsilon\theta}\right)\left(\frac{\delta}{1-\delta}\right)\frac{2\upsilon}{\gamma}\right] \end{aligned}$$

Using the simplified expression of $cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)$ and $\Sigma_\pi^{\mu_Q^{\mathbf{r}}} = \upsilon\Sigma_P^{\mathbf{r}}$, we have

$$\begin{aligned} A &= cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)^T\left(\gamma\Sigma_{\overline{Q}}^{\mathbf{r}} + \theta\Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)^{-1} cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right) \\ &= \left(\frac{\gamma}{\gamma+\upsilon\theta}\right)\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)^T(\gamma\Sigma_P^{\mathbf{r}})^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)\frac{4\upsilon^2}{\gamma^2} = \left(\frac{\gamma}{\gamma+\upsilon\theta}\right) R_{\overline{Q}}^{\mathbf{w}^d}\frac{4\upsilon^2}{\gamma^2} \\ B &= cov_\pi\left(\mu_Q^{\mathbf{r}}, R_Q^{\mathbf{w}^d}\right)^T\left(\gamma\Sigma_{\overline{Q}}^{\mathbf{r}} + \theta\Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right) = \left(\frac{\gamma}{\gamma+\upsilon\theta}\right) R_{\overline{Q}}^{\mathbf{w}^d}\frac{2\upsilon}{\gamma} \\ C &= \left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)^T\left(\gamma\Sigma_{\overline{Q}}^{\mathbf{r}} + \theta\Sigma_\pi^{\mu_Q^{\mathbf{r}}}\right)^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right) = \left(\frac{\gamma}{\gamma+\upsilon\theta}\right) R_{\overline{Q}}^{\mathbf{w}^d} \end{aligned}$$

Next, we solve

$$\begin{aligned}
& E_\pi\left(R_Q^{\mathbf{w}^d}\right) \\
= \;& E_\pi\left(\left(\mu_Q^{\mathbf{r}}-r_f\mathbf{1}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}}-r_f\mathbf{1}\right)\right) \\
= \;& E_\pi\left(\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}+\mu_{\overline{Q}}^{\mathbf{r}}-r_f\mathbf{1}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}+\mu_{\overline{Q}}^{\mathbf{r}}-r_f\mathbf{1}\right)\right) \\
= \;& E_\pi\left(\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)\right)+E_\pi\left(\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}}-r_f\mathbf{1}\right)\right)+ \\
& E_\pi\left(\left(\mu_{\overline{Q}}^{\mathbf{r}}-r_f\mathbf{1}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)\right)+E_\pi\left(\left(\mu_{\overline{Q}}^{\mathbf{r}}-r_f\mathbf{1}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}}-r_f\mathbf{1}\right)\right),
\end{aligned}$$

where the second and third terms are zero because $E_\pi\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)=E_\pi\left(\mu_Q^{\mathbf{r}}\right)-\mu_{\overline{Q}}^{\mathbf{r}}=0$. The last term is the expected delegation return under the investor's average model, $R_{\overline{Q}}^{\mathbf{w}^d}=\left(\mu_{\overline{Q}}^{\mathbf{r}}-r_f\mathbf{1}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}}-r_f\mathbf{1}\right)$. Therefore, we have

$$\begin{aligned}
E_\pi\left(R_Q^{\mathbf{w}^d}\right) &= E_\pi\left(\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)\right)+R_{\overline{Q}}^{\mathbf{w}^d} \\
&= tr\left[\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}E_\pi\left(\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)^T\right)\right]+R_{\overline{Q}}^{\mathbf{w}^d} \\
&= tr\left[\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\upsilon\Sigma_P^{\mathbf{r}}\right)\right]+R_{\overline{Q}}^{\mathbf{w}^d} \\
&= \frac{\upsilon}{\gamma}N+R_{\overline{Q}}^{\mathbf{w}^d}
\end{aligned}$$

Another way to solve $E_\pi\left(R_Q^{\mathbf{w}^d}\right)$ is to notice that under the assumption $\Sigma_\pi^{\mu_Q^{\mathbf{r}}}=\upsilon\Sigma_P^{\mathbf{r}}$, $\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)=\frac{\upsilon}{\gamma}\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)^T\left(\upsilon\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)$ is a multiple of squared normalized Gaussian variable that has a Chi-squared distribution with the degree of freedom equal to N and mean equal to $\frac{\upsilon}{\gamma}N$. Since $R_Q^{\mathbf{w}^d}$ can be decomposed,

$$\begin{aligned}
R_Q^{\mathbf{w}^d} &= \left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)+\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}}-r_f\mathbf{1}\right)+ \\
&\quad \left(\mu_{\overline{Q}}^{\mathbf{r}}-r_f\mathbf{1}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)+\left(\mu_{\overline{Q}}^{\mathbf{r}}-r_f\mathbf{1}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}}-r_f\mathbf{1}\right), \\
&= \left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)+2\left(\mu_{\overline{Q}}^{\mathbf{r}}-r_f\mathbf{1}\right)^T\left(\gamma\Sigma_P^{\mathbf{r}}\right)^{-1}\left(\mu_Q^{\mathbf{r}}-\mu_{\overline{Q}}^{\mathbf{r}}\right)+R_{\overline{Q}}^{\mathbf{w}^d},
\end{aligned}$$

and the second term has zero mean, we have

$$R_Q^{\mathbf{w}^d}=\frac{\upsilon}{\gamma}N+R_{\overline{Q}}^{\mathbf{w}^d}.$$

Similarly, to solve $\sigma_\pi^2\left(R_Q^{\mathbf{w}^d}\right)$, we also use the decomposition of $R_Q^{\mathbf{w}^d}$. The first term has a Chi-squared distribution, so its variance is

$$\sigma^2_\pi\left(\left(\mu^{\mathbf{r}}_{Q}-\mu^{\mathbf{r}}_{\overline{Q}}\right)^T\left(\gamma\Sigma^{\mathbf{r}}_{P}\right)^{-1}\left(\mu^{\mathbf{r}}_{Q}-\mu^{\mathbf{r}}_{\overline{Q}}\right)\right)=\frac{\upsilon^2}{\gamma^2}2N.$$

The second term is a linear transformation of Gaussian variable $\left(\mu^{\mathbf{r}}_{Q}-\mu^{\mathbf{r}}_{\overline{Q}}\right)$ that has variance equal to $\Sigma^{\mu^{\mathbf{r}}_{Q}}_{\pi}=\upsilon\Sigma^{\mathbf{r}}_{P}$, so the second term's variance is

$$\sigma^2_\pi\left(\left(\mu^{\mathbf{r}}_{Q}-\mu^{\mathbf{r}}_{\overline{Q}}\right)\right)=2\left(\mu^{\mathbf{r}}_{\overline{Q}}-r_f\mathbf{1}\right)^T\left(\gamma\Sigma^{\mathbf{r}}_{P}\right)^{-1}\upsilon\Sigma^{\mathbf{r}}_{P}\left(\gamma\Sigma^{\mathbf{r}}_{P}\right)^{-1}\left(\mu^{\mathbf{r}}_{\overline{Q}}-r_f\mathbf{1}\right)2=4\frac{\upsilon}{\gamma}R^{\mathbf{w}^d}_{\overline{Q}}.$$

The third term is a constant, the expected delegation return under the average model, so its' variance is zero. To solve $\sigma^2_\pi\left(R^{\mathbf{w}^d}_{Q}\right)$, we still need the covariance between the Chi-squared first component of $R^{\mathbf{w}^d}_{Q}$ and the Gaussian second component of $R^{\mathbf{w}^d}_{Q}$. First, we notice that the first moment of their product is zero:

$$\begin{aligned}
&E_\pi\left(\left(\mu^{\mathbf{r}}_{Q}-\mu^{\mathbf{r}}_{\overline{Q}}\right)^T\left(\gamma\Sigma^{\mathbf{r}}_{P}\right)^{-1}\left(\mu^{\mathbf{r}}_{Q}-\mu^{\mathbf{r}}_{\overline{Q}}\right)2\left(\mu^{\mathbf{r}}_{\overline{Q}}-r_f\mathbf{1}\right)^T\left(\gamma\Sigma^{\mathbf{r}}_{P}\right)^{-1}\left(\mu^{\mathbf{r}}_{Q}-\mu^{\mathbf{r}}_{\overline{Q}}\right)\right)\\
=\;&2\left(\mu^{\mathbf{r}}_{\overline{Q}}-r_f\mathbf{1}\right)^T\left(\gamma\Sigma^{\mathbf{r}}_{P}\right)^{-1}E_\pi\left(\left(\mu^{\mathbf{r}}_{Q}-\mu^{\mathbf{r}}_{\overline{Q}}\right)^T\left(\gamma\Sigma^{\mathbf{r}}_{P}\right)^{-1}\left(\mu^{\mathbf{r}}_{Q}-\mu^{\mathbf{r}}_{\overline{Q}}\right)\left(\mu^{\mathbf{r}}_{Q}-\mu^{\mathbf{r}}_{\overline{Q}}\right)\right)\\
=\;&\frac{2\upsilon}{\gamma}\left(\mu^{\mathbf{r}}_{\overline{Q}}-r_f\mathbf{1}\right)^T\left(\gamma\Sigma^{\mathbf{r}}_{P}\right)^{-1}E_\pi\left(\left(\mu^{\mathbf{r}}_{Q}-\mu^{\mathbf{r}}_{\overline{Q}}\right)^T\left(\upsilon\Sigma^{\mathbf{r}}_{P}\right)^{-1}\left(\mu^{\mathbf{r}}_{Q}-\mu^{\mathbf{r}}_{\overline{Q}}\right)\left(\mu^{\mathbf{r}}_{Q}-\mu^{\mathbf{r}}_{\overline{Q}}\right)\right),
\end{aligned}$$

which is equal to zero because by Isserlis' theorem, the expectation of three zero-mean multivariate normal variables is zero. Since the second component has zero mean, the product of its and the first term's first moments is also zero. Therefore, the covariance between the first and second components of $R^{\mathbf{w}^d}_{Q}$ is zero. To sum up,

$$\sigma^2_\pi\left(R^{\mathbf{w}^d}_{Q}\right)=\frac{\upsilon^2}{\gamma^2}2N+4\frac{\upsilon}{\gamma}R^{\mathbf{w}^d}_{\overline{Q}}.$$

Substitute the solutions of A, B, C, $E_\pi\left(R^{\mathbf{w}^d}_{Q}\right)$, and $\sigma^2_\pi\left(R^{\mathbf{w}^d}_{Q}\right)$ into the optimal δ, we have

$$\begin{aligned}
\delta&=\frac{E_\pi\left(R^{\mathbf{w}^{d}(q)}_{Q}\right)-(\theta+\gamma)B-C-\psi}{E_\pi\left(R^{\mathbf{w}^{d}(q)}_{Q}\right)+\theta\sigma^2_\pi\left(R^{\mathbf{w}^{d}(q)}_{Q}\right)-(\theta+\gamma)^2A-2(\theta+\gamma)B-C}\\
&=\frac{\frac{\upsilon}{\gamma}N-\psi+\left[1-\frac{\gamma}{\gamma+\upsilon\theta}\left(\frac{2\upsilon(\theta+\gamma)}{\gamma}+1\right)\right]R^{\mathbf{w}^d}_{\overline{Q}}}{\frac{\upsilon}{\gamma}N+\theta\frac{\upsilon^2}{\gamma^2}2N+\left[1+4\frac{\theta\upsilon}{\gamma}-\frac{\gamma}{\gamma+\upsilon\theta}\left(\frac{2\upsilon(\theta+\gamma)}{\gamma}+1\right)^2\right]R^{\mathbf{w}^d}_{\overline{Q}}}
\end{aligned}$$

Comparative statics. The investor's portfolio is

$$\mathbf{w}^o = (\Sigma_P^{\mathbf{r}})^{-1}\left(\mu_{\overline{Q}}^{\mathbf{r}} - r_f\mathbf{1}\right)\left[\frac{1}{\gamma+\upsilon\theta} - \frac{\upsilon\gamma+\upsilon\theta}{\gamma+\upsilon\theta}\frac{\delta}{1-\delta}\frac{2}{\gamma}\right]$$

The optimal delegation decision is

$$\delta = \frac{\frac{\upsilon}{\gamma}N - \psi + \left[1 - \frac{\gamma}{\gamma+\upsilon\theta}\left(\frac{2\upsilon(\theta+\gamma)}{\gamma}+1\right)\right]R_{\overline{Q}}^{\mathbf{w}^d}}{\left(1+2\frac{\theta\upsilon}{\gamma}\right)\frac{\upsilon}{\gamma}N + \left[1+4\frac{\theta\upsilon}{\gamma} - \frac{\gamma}{\gamma+\upsilon\theta}\left(\frac{2\upsilon(\theta+\gamma)}{\gamma}+1\right)^2\right]R_{\overline{Q}}^{\mathbf{w}^d}}$$

Under the three special conditions, we prove the following results of comparative statics:

- $\frac{\partial\delta}{\partial N} > 0$. Proof: As long as N is larger than the expected return of $\mathbf{w}^d\left(\overline{Q}\right)$ under $\overline{Q}$. Since $\delta \in (0,1)$, δ increases in N:

$$\frac{2\frac{\theta\upsilon}{\gamma}\frac{\upsilon}{\gamma}N + \psi + \left[4\frac{\theta\upsilon}{\gamma} - \frac{2\upsilon(\theta+\gamma)}{\gamma+\upsilon\theta}\left(\frac{2\upsilon(\theta+\gamma)}{\gamma}+1\right)\right]R_{\overline{Q}}^{\mathbf{w}^d}}{\left(1+2\frac{\theta\upsilon}{\gamma}\right)\frac{\upsilon}{\gamma}N + \left[1+4\frac{\theta\upsilon}{\gamma} - \frac{\gamma}{\gamma+\upsilon\theta}\left(\frac{2\upsilon(\theta+\gamma)}{\gamma}+1\right)^2\right]R_{\overline{Q}}^{\mathbf{w}^d}} > 0$$

- $\frac{\partial\delta}{\partial\upsilon} < 0$, $\frac{\partial\delta}{\partial\theta} < 0$ and $\frac{\partial\delta}{\partial\gamma} > 0$. As long as N is large enough, δ decreases in υ and θ, and increases in γ:

$$\delta = \frac{1 + \frac{\gamma}{\upsilon N}\left(R_{\overline{Q}}^{\mathbf{w}^d} - \frac{\gamma}{\gamma+\upsilon\theta}\left(\frac{2\upsilon(\theta+\gamma)}{\gamma}+1\right)R_{\overline{Q}}^{\mathbf{w}^d} - \psi\right)}{\left(1+2\frac{\theta\upsilon}{\gamma}\right) + \frac{\gamma R_{\overline{Q}}^{\mathbf{w}^d}}{\upsilon N}\left(1+4\frac{\theta\upsilon}{\gamma} - \frac{\gamma}{\gamma+\upsilon\theta}\left(\frac{2\upsilon(\theta+\gamma)}{\gamma}+1\right)^2\right)}$$

- $\frac{\partial\mathbf{w}^o}{\partial\upsilon} < 0$, $\frac{\partial\mathbf{w}^o}{\partial\theta} < 0$ and $\frac{\partial\mathbf{w}^o}{\partial\gamma} < 0$, conditional on δ. Proof: Note that $\frac{\upsilon\gamma+\upsilon\theta}{\gamma+\upsilon\theta} = \frac{\upsilon\gamma+\upsilon\theta}{\gamma+\upsilon\theta} < 1$ as long as $\upsilon < 1$. Hence given δ, $\frac{1}{\gamma+\upsilon\theta} - \frac{\upsilon\gamma+\upsilon\theta}{\gamma+\upsilon\theta}\frac{\delta}{1-\delta}\frac{2}{\gamma}$ decreases in υ and θ.

$$\frac{\partial\frac{\gamma+\theta}{\gamma+\upsilon\theta}\frac{1}{\gamma}}{\partial\gamma} = \frac{-(1-\upsilon)\theta}{(\gamma+\upsilon\theta)^2}\frac{1}{\gamma} - \frac{1}{\gamma^2}\left(\frac{\gamma+\theta}{\gamma+\upsilon\theta}\right) < 0$$

as long as $\upsilon < 1$.

- $\frac{\partial\mathbf{w}^o}{\partial\upsilon} > 0$, $\frac{\partial\mathbf{w}^o}{\partial\theta} > 0$, $\frac{\partial\mathbf{w}^o}{\partial\gamma} < 0$ and $\frac{\partial\left[\frac{1}{\gamma+\upsilon\theta} - \frac{\upsilon\gamma+\upsilon\theta}{\gamma+\upsilon\theta}\frac{\delta}{1-\delta}\frac{2}{\gamma}\right]\gamma}{\gamma} < 0$. Proof: For large N, $\frac{\upsilon\theta+\upsilon\gamma}{\gamma+\upsilon\theta}\left(\frac{\delta}{1-\delta}\right) \approx \frac{1+\gamma/\theta}{1+\upsilon\theta/\gamma}\frac{1}{2}$, so,

$$\frac{\partial\left[\frac{1}{\gamma+\upsilon\theta} - \frac{\upsilon\gamma+\upsilon\theta}{\gamma+\upsilon\theta}\frac{\delta}{1-\delta}\frac{2}{\gamma}\right]}{\partial\theta} \approx \frac{2\frac{\gamma^2}{\theta^2} + 2\frac{2\upsilon\gamma}{\theta} + \upsilon}{(\gamma+\upsilon\theta)^2} > 0$$

$$\frac{\partial\left[\frac{1}{\gamma+\upsilon\theta}-\frac{\upsilon\gamma+\upsilon\theta}{\gamma+\upsilon\theta}\frac{\delta}{1-\delta}\frac{2}{\gamma}\right]}{\partial\upsilon}\approx\frac{\theta+2\gamma}{\left(\gamma+\upsilon\theta\right)^{2}}>0$$

$$\frac{\partial\left[\frac{1}{\gamma+\upsilon\theta}-\frac{\upsilon\gamma+\upsilon\theta}{\gamma+\upsilon\theta}\frac{\delta}{1-\delta}\frac{2}{\gamma}\right]}{\partial\gamma}\approx\frac{-\upsilon}{\left(\gamma+\upsilon\theta\right)^{2}}<0$$

$$\frac{\partial\left[\frac{1}{\gamma+\upsilon\theta}-\frac{\upsilon\gamma+\upsilon\theta}{\gamma+\upsilon\theta}\frac{\delta}{1-\delta}\frac{2}{\gamma}\right]\gamma}{\partial\gamma}\approx\frac{-\frac{\gamma^{2}}{\theta}-2\gamma\upsilon}{\left(\gamma+\upsilon\theta\right)^{2}}<0$$

- When, $N<\frac{1}{\upsilon}\left[\left(\gamma+\theta+\frac{\gamma}{2\upsilon}\right)\psi+\left(\theta-\gamma\right)R_{\overline{Q}}^{\mathbf{w}^{d}}\right]$, $\mathbf{w}^{o}\geq\mathbf{0}$ if and only if $\mu_{\overline{Q}}^{\mathbf{r}}>r_{f}\mathbf{1}$. Proof:

$$\begin{aligned}\frac{1}{\gamma+\upsilon\theta} &> \frac{\upsilon\gamma+\upsilon\theta}{\gamma+\upsilon\theta}\frac{\delta}{1-\delta}\frac{2}{\gamma}\\ &\Leftrightarrow\\ N &< \frac{1}{\upsilon}\left[\left(\gamma+\theta+\frac{\gamma}{2\upsilon}\right)\psi+\left(\theta-\gamma\right)R_{\overline{Q}}^{\mathbf{w}^{d}}\right]\end{aligned}$$

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