# Tokenomics: Dynamic Adoption and Valuation*

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*PRELIMINARY. COMMENTS WELCOME.*

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**Abstract**

We provide a dynamic asset-pricing model of (crypto-) tokens on (blockchain-based) platforms. Tokens intermediate peer-to-peer transactions, and their trading creates inter-temporal complementarity among users and generates a feedback loop between token valuation and adoption. Consequently, tokens capitalize future platform growth, accelerate adoption, and reduce user-base volatility. Equilibrium token price increases non-linearly in platform productivity, user heterogeneity, and endogenous network size. Consistent with evidence, the model produces explosive growth of user base after an initial period of dormant adoption, accompanied by a run-up of token price volatility.

**JEL Classification: C73, F43, E42, L86**

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# 1 Introduction

Blockchain-based cryptocurrencies and tokens have taken the world by storm. According to CoinMarketCap.com, the entire cryptocurrency market capitalization has also grown from around US$20 billion to around US$600 billion over last year, with active trading and uses;[1] virtually unknown a year ago, ICOs are also now more celebrated and debated than the conventional IPOs, raising 3.5 billion in more than 200 ICOs in 2017 alone, according to CoinSchedule. However, it is far from clear if cryptocurrency would completely escape regulation or replace conventional money, especially given the large volatility involved. There lacks no critics of the development of cryptocurrency, at least Bitcoin, in both the industry and academia.[2] ICOs are also facing quagmires regarding its legitimacy and distinction from security issuance.[3] In the recent hearing on Capital Markets, Securities, and Investment Wednesday, March 14, 2018, the regulators appear rather divided, if not outright "confused", on the future of cryptocurrencies, digital currencies, ICOs, and Blockchain development.

In order to draw a line between reckless speculation and financial innovation, and understand whether tokens should be regulated like securities, it is important to first understand how cryptocurrency or tokens derive their value. What fundamentals drive their pricing and volatility? How do they interact with the endogenous user adoption and the development of the blockchain technology and virtual economy? What are the new economic insights about asset pricing we can gain by analyzing the value of cryptocurrency or tokens? What can we learn about the valuation of platform or user base in general?

Motivated by these questions and the debates in both industry and academia, we develop the first dynamic model of virtual economy with endogenous user adoption dynamics and a native cryptocurrency/token (henceforth generically referred to as "token") that facilitates transactions and business operations (e.g., smart contracting) on a blockchain. We anchor token valuation on the fundamental productivity of the blockchain technology, and demonstrate how tokens derive value as an exchangeable asset with limited supply that users hold to derive utility. Our model highlights that tokens capitalize user-base growth in the future: agents' expectation of technological progress and popularity of the blockchain system translates into expected token price appreciation, which makes tokens an attractive investment and induces more agents to hold tokens and join the ecosystem.

[1] For example, many retailers in Japan already accept Bitcoins (e.g., Holden and Subrahmanyam (2017)).

[2] Rogoff (2017) and Shiller (2017) are notable representatives, although Cochrane argues otherwise in *The Grumpy Economist*, Nov 20, 2017.

[3] See, for example, "Token Resistance," The Economist, November 11th, 2017.

Specifically, we consider a continuous-time economy with a unit mass of agents who differ in their needs to conduct transactions on the blockchain. We broadly interpret transaction as including not only typical money transfer (e.g., on the bitcoin blockchain) but also signing smart contracts (e.g., on the Ethereum blockchain). Accordingly, we model agents' gain from blockchain transaction as a flow utility that depends on the current state of blockchain platform ("productivity shock" broadly interpreted), agent-specific transaction needs, and very importantly, the size of blockchain community. The larger the community is, the more surplus can be realized through trades among agents on the blockchain. In our model, agents make a two-step decision: (1) whether to incur a participation cost to meet potential trade counterparties; (2) how many tokens to hold, which depends on both blockchain trade surplus and the expected future token price. And, through the impact on community size, agents' decision exerts externalities on each other. An expectation of price appreciation leads to stronger demand for tokens, and more agents joining the blockchain community.

The very requirement that agents must hold tokens to conduct transactions is consistent with many existing applications: Tokens are the required medium of exchange for transactions and business operations on blockchain platforms, either by protocol design or by the fact that they offer higher convenience yield relative to alternative currencies. In fact, contrary to the general perception, most "utility tokens" issued through ICOs are not "corporate coupons" that are used to redeem products or services from the issuing company, but instead are the required means of payment for products and services from other blockchain users.

Here we highlight that the benefits of using such tokens increases in the size of blockchain user base. As a result, token price reflects the future growth of the community, and becomes higher, if the expected user-base growth is stronger. Taking a step back, when the platform technology is forecast to improve, inducing more agents to join the community, the consequent expectation of token price appreciation feeds into agents current decision to join the community and hold tokens. Therefore, the existence of tokens as a native currency not only serves for technological purposes as practitioners argue, but more importantly, advances the growth of user base by reflecting agents' expectation of future technological progress, larger user base, and higher token price. The model equilibrium features an inter-temporal complementarity of user base – expectation of more users in the future feeds into more users today. Nevertheless, we show there exists a unique non-degenerate Markov equilibrium under standard regularity conditions of continuous-time models.

We compare blockchain ecosystems with and without tokens, and show that those with tokens see their user base grow faster when the underlying technology is expected to improve over time, resulting in a higher welfare. Moreover, introducing tokens reduces the volatility of user base, making the size of it less sensitive to technology shock. The intuition is straightforward: agents' decision to participate depends on their expectation of future token price appreciation. Consider a temporary negative shock that reduces the trade surplus that can be realized among agents, and thus, reduces the user base. The shock impact on user base is alleviated because agents now expect stronger token price appreciation. A smaller user base now implies a stronger growth of user base in the future, because there are more agents off the blockchain that will be brought on board.

Akin to many equilibrium models that feature interaction between financial markets and the real economy, the financial side of our model is the endogenous price of tokens, while the real side is the user base that determines the benefits of individual agents who trade on the blockchain. Token price affects user base through agents' expectation of the future price appreciation, but at the same time, user base affects token price through agents' demand. We show that this feedback loop generates a rich joint dynamics of token price and user base, and calibrate our model to match such pattern in data.

Our model sheds light on the pricing of means of payment in peer-to-peer networks that are the defining features of many virtual economies.[4] Many private digital currencies have been set up in association with platforms whether payment-focused or not: Linden dollar for the game Second Life, WoW Gold for the game World of Warcraft, Facebook Credits, Q-coins for Tencent QQ, Amazon coins, to name a few.[5] The Blockchain technology gives platforms unprecedented flexibility and commitment power in introducing native currencies and designing their attributes, yet we lack a valuation framework. Our model offers a pricing formula and reveals how introducing native currencies benefit users and accelerate their adoption. Our framework potentially applies to a wide range of applications, such as email protocols and online social network. More generally, our framework can be applied to asset pricing and macro models with network externality. When more agents invest in certain assets either now or in future, everyone benefits from the increase in network size.

[4] To be precise, the networks in question should be viewed as complete networks with economy of scale, which is different from incomplete networks that many recent studies focus on (e.g., Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012)).

[5] Even before the heated debate on cryptocurrencies, economists, and commentators were already raising questions such as "Could a gigantic nonsovereign like Facebook someday launch a real currency to compete with the dollar, euro, yen and the like?" (Yglesias (2012)). Gans and Halaburda (2015) provides an insightful introduction on how payment systems and platforms are related.

**Literature Review.** Our paper first contributes to the emerging literature on FinTech, especially on blockchains and cryptocurrencies. The concepts of digital currency and distributed ledger have been separately developed and Nakamoto's innovation lies in combining them to enable large-scale application (Narayanan and Clark (2017)). Nakamoto (2008) highlights that embedding a native currency into a blockchain system helps incentivize record-keepers (e.g., miners in protocols using proof-of-work) of decentralized consensus, which in turn helps preventing double-spending. Our paper emphasizes the incentives of users, and shows that introducing a native currency or token accelerates user adoption, which feeds back to cryptocurrency/token valuation.

Among studies on the application and economic impact of the blockchain technology, Harvey (2016) briefly surveys the mechanics and applications of crypto-finance, especially Bitcoin.[6] Yermack (2017) evaluates the potential impacts of the blockchain technology on corporate governance. Cong and He (2018) emphasize information distribution in generating decentralized consensus, with implications on industrial organization. Several studies analyze crypto-currency mining games (e.g., Eyal and Sirer (2014), and Biais, Bisiere, Bouvard, and Casamatta (2017)) and miners' compensation and organization (e.g., Easley, O'Hara, and Basu (2017), Huberman, Leshno, and Moallemi (2017), and Cong, He, and Li (2018)).

Our paper takes the technological aspects as given, and focuses on the valuation of cryptocurrencies and tokens under endogenous user adoption in a *dynamic* framework that highlights inter-temporal feedback effects. In contrast, other models in the literature are static. A contemporaneous paper by Pagnotta and Buraschi (2018) studies the asset pricing implications of user-base externality in a static setting, with a focus on Bitcoin and miners' incentives but takes user base as exogenous. Gans and Halaburda (2015) is among the earliest studies on platform-specific virtual currencies and users' network effects. Ciaian, Rajcaniova, and Kancs (2016) test quantity theory of money using Bitcoin data without modeling agents' optimal decision to use Bitcoins. Fernández-Villaverde and Sanches (2016) and Gandal and Halaburda (2014) consider the competition among alternative cryptocurrencies. Closer to our paper is Athey, Parashkevov, Sarukkai, and Xia (2016) that emphasizes agents' dynamic learning on a binary technology quality and decision to use bitcoins for money transfer, but

[6] For a brief introduction to blockchains and ICOs, see Nanda, White, and Tuzikov. (2017a,b). Other papers on more specialized applications include: Catalini and Gans (2016) point out the blockchain technology can reduce the cost of verification and the cost of networking. Malinova and Park (2016) study the design of the mapping between identifiers and end-investors and the degree of transparency of holdings in a blockchain-based marketplace. Khapko and Zoican (2017) argue that blockchain allows for flexible settlement of trades, and the optimal time-to-settle trades off search costs and counter-party risk, creating vertical differentiation.

does not model blockchain productivity and user-base externality.

Several contemporaneous studies analyze cryptocurrencies in the context of initial coin offerings (ICOs). In a two-period setting, Sockin and Xiong (2018) study how households first purchase a indivisible cryptocurrency which serves as membership certificate that enables them to match and trade later. Li and Mann (2018) demonstrate that staged coin offerings mitigate coordination issues. Catalini and Gans (2018) study ICOs with tokens as coupons for redeeming products and services from the issuing company. Both Li and Mann (2018) and Catalini and Gans (2018) further argue that ICOs help aggregate dispersed private information.

Instead of focusing on information aggregation or financing through coin offerings, we emphasize the role of cryptocurrencies in accelerating user adoption and crypto-tokens as means of payment in decentralized virtual economy, which fits most token white papers in practice thus far.[7] We differ also in characterizing heterogeneous agents' endogenous and divisible holdings of tokens, and considering both adoption complementarity and inter-temporal feedback effect of tokens. While other studies typically focus on equilibrium multiplicity, we are the first to jointly pin down the value of tokens and user adoption.

Also related are discussions on the design of cryptocurrencies/tokens and platforms. Gans and Halaburda (2015) link research on platforms and payments systems literature Chiu and Wong (2015) take a mechanism design approach to discuss how e-money helps implement constrained efficient allocations. Chiu and Koeppl (2017) discuss how cryptocurrency systems can improve efficiency through alterntiave designs.Balvers and McDonald (2017) discuss a blockchain-based global currency that can be used to achieve an ideal currency stable in terms of purchasing power. By providing a dynamic pricing framework, we directly tie the protocol-based design on supplies to cryptocurrency valuation and user adoption, enabling us to evaluate various designs.

We organize the remainder of the article as follows. Section 2 provides the institutional background on crypto-currency and crypto-tokens. Section 3 sets up the model and characterizes the dynamic equilibrium. Section 4 performs quantitative analysis and discusses model implications. Section 5 contains extensions and further discussions. Section 6 concludes.

[7] In a companion study, Cong, Li, and Wang (2018), we embed our pricing framework into ICOs as a financing innovation, emphasizing tokens' role as means of payments (as opposed to membership certificate or product/service coupons) among users and accelerator of technology/platform development when held by dispersed contributors.

## 2 Institutional Background

In this section, we briefly introduce the development of the blockchain technology, and then clarify various concepts associated with cryptocurrency, which are not mutually exclusive and are constantly evolving. Importantly, we highlight two salient features shared among the majority of cryptocurrencies, crypto-tokens, and platform currencies: first, they are used as means of payment by design ("monetary embedding"); second, their typical application scenarios exhibit some forms of network effect ("user-base externality"). We also briefly describe the asset category they belong to. Readers already familiar with the institutional background and these features may wish to skip to the model directly.

**Blockchain, Cryptocurrency, and Token**. The advances in FinTech and sharing economy is largely driven by the increasing preference for forming peer-to-peer connections that are instantaneous and open, which is transforming how people interact, work together, consume and produce. Yet financial systems are rather centralized, arranged around a set of key players such as asset managers, banks, and payment and settlement institutions. Blockchain-based applications are part of an attempt to create a peer-to-peer financial architecture, reorganizing the society into a set of decentralized networks of human interactions. By providing decentralized consensus, blockchains allow peers unknown to and distant from one another to interact, transact, and contract without relying a centralized trusted third party. The technology is believed to potentially avoid single point of failure and even reduce concentration of market power, but still face many challenging issues.[8]

Even though not always necessarily required, a majority of blockchain applications entail the use of cryptocurrencies and tokens. There is a lack of clarity, if not general confusion, on the myriad of terms and references such as cryptocurrencies, altcoins, appcoins, tokens, etc. A lot of these concepts are not mutually exclusive and are starting to be used interchangeably.

Cryptocurrencies are cryptography-secured digital or virtual currencies. Bitcoin rep-

[8]Although Bank of England governor Mark Carney dismissed Bitcoin as an alternative currency, he recognized that the blockchain technology benefits data management by improving resilience by "eliminating central points of failure" and enhancing transparency and auditability while expanding what he called the use of "straight-through processes" including with smart contracts. In particular, "Crypto-assets help point the way to the future of money". See, e.g., beat.10ztalk.com. For various applications of the technology, we refer the readers to Harvey (2016) and Yermack (2017), and for smart contracting, Cong and He (2018).

resents the first widely-adopted decentralized cryptocurrency, and popularized the concept. Besides Bitcoin, over 1000 different "altcoins" (stand for for alternative cryptocurrency coins, alternative to Bitcoin) have been introduced over the past few years and many central banks are actively exploring the area for retail and payment systems.[9] Many altcoins such as Litecoin and Dogecoin are variants (forks) from Bitcoin, with mofidications to the original open-sourced protocol to enable new features. Others such as Ethereum and Ripple created their own Blockchain and protocol to support the native currency. Cryptocurrencies are typically regarded as payment-focused and primarily associated with their own independent blockchain. In these payment and settlement applications as exemplified by Bitcoin and Ripple, cryptocurrencies obviously act as means of payment on their respective blockchain platforms.

Meanwhile, Blockchain-based crypto-tokens have also gained popularity. In what is known as Initial Coin Offerings (ICO), entrepreneurs sell "tokens" or "AppCoins" to dispersed investors around the globe.[10] Tokens are representations of claims on issuers' cashflow, rights to redeem issuers' products and services, or means of payment among blockchain users. They usually operate on top of an existing blockchain infrastructure to facilitates the creation of decentralized applications.[11]

Some tokens derive their value from the company's future cashflow, and thus, serve a function similar to securities (thus termed "security tokens"). They are well-understood and are not our focus. The vast majority of ICOs that launched in 2016 and 2017 were "utility tokens", which include many of the highest-profile projects: Filecoin, Golem, 0x, Civic, Raiden, Basic Attention Token (BAT), and more. Many media reporters, practitioners, regulators, and even academics often analogize utility tokens to "corporate coupons", which allow consumers to redeem products or services from the service provider. Although some

[9]For example, People's Bank of China aims to develop a digital currency system; Bank of Canada and Singapore Monetary Authority use blockchain for interbank payment systems; Deutsche Bundesbank works on prototype of blockchain-based settlement systems for financial assets; in a controversial move, the government of Venezuela became the first federal government to issue digital currency and announced on Feb 20, 2018 the presale of its "petro" cryptocurrency — an oil-backed token as a form of legal tender that can be used to pay taxes, fees and other public needs.

[10]While the first ICO in 2013 raised a meager $ 500k and sporadic activities over the next two years. 2016 saw 46 ICOs raising about $ 100m and according to CoinSchedule, in 2017 there were 235 Initial Coin Offerings. The year-end totals came in over $3 billion raised in ICO. In August, 2017, OmiseGO (OMG) and Qtum passed a US$1 billion market cap today, according to coinmarketcap.com, to become the first ERC20 tokens built on the Ethereum network and sold via an ICO to reach the unicorn status.

[11]By "on top of" a blockchain, we mean that one can use smart contract templates, for example on the Ethereum or Waves platform, to create tokens for particular applications, without having to create or modify codes from other blockchain protocols.

tokens are indeed corporate coupons, it is thus far neglected that the majority of them are not.[12]

As we illustrate using some of the aforementioned tokens shortly, utility tokens are often the required means of payment among users for certain products or services, or represent certain opportunities to provide blockchain services for profit as in the case of "stake tokens". Precisely here lies the key innovation of the blockchain technology: allowing peer-to-peer interactions in decentralized networks, as opposed to designing and auctioning coupons issued by centralized product/service providers – an old phenomenon economists understand relatively well (e.g., pay-in-kind crowdfunding).

In this paper, we focus on the common features shared by cryptocurrencies and utility tokens that serve essentially the role of means of payment among blockchain users. We thus use "tokens", "cryptocurrencies", etc., interchangeably and often collectively refer to them as "tokens". Next, we highlight these unique features of the blockchain technology that distinguishes the economics of introducing and valuing tokens from what we already know in the literature of monetary economics and asset pricing.

**Monetary Embedding.** Many blockchain-based decentralized networks introduce native currencies – a phenomenon we call "monetary embedding". In the following, we elaborate on the rationales behind such phenomenon and relate them to the issue of money velocity, setting the stage for our formal analysis in Section 3.

First, in the virtual economy, potential users are likely from around the globe, using fiat money issued by and subject to specific countries' legal and economic influences. Transacting in a uniform currency is simply more convenient, free from the transaction costs of currency exchange. For example, it is cheaper to make international payments and settlements using Ripples (XRP) on the Ripple network. Even though Ethereum platform allows other AppCoins and cryptocurrencies (provided that they are ERC-20 compatible), many transactions and fundraising activities are still carried out using Ethers (ETH) because of its convenience and popularity (i.e., widely accepted by Ethereum users).

Second, from a theoretical perspective, it is advantageous to adopt a standard unit of account in the ecosystem because it mitigates the risks of asset-liability mismatches when they are denominated in different units of account (Doepke and Schneider (2017)). This is particularly relevant on a blockchain platform designed for smart contracting. Some argue

[12] We should be very clear about this distinction because not only the valuation framework differs, but legal and regulatory implications differ as well.

that the lack of trust in an online space, very much due to the anonymity of participants, implies that trade has to be quid pro quo, so a means of payment is required (Kiyotaki and Wright (1989)).

Then why not just use US dollars or other existing currencies as settlement media? This leads to the third rationale: native currency helps to incentivize miners, validators, and users to contribute to the stability, functionality (provision of decentralized consensus), and prosperity of the ecosystem (Nakamoto (2008)). For example, for blockchain applications where decentralized consensus is achieved through the mechanism of "proof-of-stake", the ownership of native currency entitles platform users to be the consensus generator/recorder; for blockchains relying on "proof-of-work" such as Bitcoins and Filecoins, native tokens are used to reward miners for block creations in the consensus processes; moreover, to profit from providing validation services, OmiseGo tokens (OMG) are required as proof of stakes on the OmiseGo blockchain. If a blockchain application is developed without a native currency, then the incentive of users is no longer directly linked to the platform in question. Practitioners are very well aware of this issue, as Strategic Coin explains in its BAT token launch research report.[13]

Fourth, introducing native currency allows the issuer to collect seigniorage, especially through initial offer coin offering (ICO). In contrast to sovereigns who cannot easily commit to a money supply rule, blockchain developers can commit to an algorithmic rule of coin supply, and therefore, create certain degree of scarcity. Provided that users need to hold coins to transact on the platform, a positive coin price can arise in an equilibrium, and such value is collected by the developers at ICO, reflecting a form of monopoly rent – the fact that users can only conduct some activities on a particular blockchain platform translates into a high price of its native currency, and more ICO revenues to the developer.

These rationales motivate us to focus on platforms with native coins. But let us take a step back and ask why cryptocurrency may have a determinant value in the first place. In principle, if one wants to transact on a blockchain platform, one can exchange dollars for its native currency, and make a transfer on the blockchain, and then immediately, the payee may exchange the native currency back into dollars. If the whole process happens instantaneously, i.e., the velocity of native currency is infinite, then there does not exist a net demand for native currency, so there exists an equilibrium of zero dollar price and equilibria with any

[13]BAT serves as a medium exchange between users, advertisers, and publishers who participate in the Brave browser ecosystem. Advertisers purchase ads using BAT tokens, which are then distributed among both publishers and browser users as compensation for hosting the ads and viewing them, respectively.

positive level of price of native currency. Therefore, we need to pin down a positive demand, so that with a algorithmically controlled supply, token price can be determined. This brings us to the second aspect of monetary embedding: agents actually need to hold the coins to profit from on-chain activities. This is indeed the case in practice for at least three reasons.

First, a demand may arise because decentralized miners or service providers ("keepers") may have to hold the native currencies to earn the right to serve the system. Proof-of-Stake protocols typically fall in this category. These tokens are sometimes referred to as work tokens or staking tokens, and notable implementations include Keep (off-chain private computation), Filecoin (distributed file storage), Truebit (off-chain computation), Livepeer (distributed video encoding), and Gems (decentralized mechanical Turk). To enforce some sort of mechanism to penalize workers who fail to perform their job to some pre-specified standard, work tokens have to be held. For example, in Filecoin, service providers contractually commit to storing some data with 24/7 access and some minimum bandwidth guarantee for a specified period of time. During the contract term, service providers must "escrow" some number of Filecoin, which can be automatically slashed (taken away) should they fail to perform the service.

Second, blockchains enable the use of smart contracts—digital contracts allowing terms contingent on decentralized consensus that are typically self-enforcing and tamper-proof through automated execution. Smart contracts typically automate transfer of tokens once certain contingencies are triggered (e.g., default), which in turn requires that the corresponding amount of cryptocurrencies must be "escrowed" during the episode that the contingencies can be triggered.[14] In other words, agents hold cryptocurrency as collateral.[15]

Third, because the generation of decentralized consensus takes time, there is a technical limit on how quickly transactions can be validated and recorded. While many protocols such as the Lightening Network and Ethereum process transactions significantly faster than Bitcoin (seconds versus 10-11 minutes), the decentralized nature of the validation means it always takes some amount of time to ensure robustness and synchronization of the consensus. During such confirmation time, cryptocurrency cannot be liquidated by either party of the transfer.

[14] Balvers and McDonald (2017) also argues that automated collateral in terms of tokens can help stabilize the purchasing power of cryptocurrency, a point very related to our emphasize on a positive crytocurrency demand.

[15] While this is similar to the traditional third-party escrow accounts, what it implies is that the coins are locked up with at least one contracting party. Arguably, blockchains and smart contracting also help solve issues related to non-exclusive contracts, through the use of native tokens to fully keep track of the collateral and escrow.

In line with these considerations, our model assumes that agents, who want to enjoy a trade surplus on the blockchain, hold its cryptocurrency for at least an instant. This holding period is important. No matter how short it is, it exposes the owner of cryptocurrency to the fluctuation of its price, so that users of the platform care not only the surplus from conducting trade with peer users but also the future coin price, which in turn depends on further user base. Next, we discuss user-base externality, another key feature of decentralized network that our model captures.

**User-Base Externality.** User-base externality has been well recognized as one of the defining features of P2P platforms, sharing economy, and various decentralized systems. When more people join the platform, individuals enjoy more surplus through interacting with other users because it is easier to find a trade partner. "Trade" here can be very general, encompassing selling products and services and signing a long-term financing contracts. The utility of using cryptocurrencies and crypto-tokens obviously goes up when more people use the blockchain platform.[16] Moreover, UnikoinGold on Unikrn (decentralized token for betting on e-sports and gambling) and Augur (decentralized prediction market) are examples showing that achieving a critical mass is crucial in platform business (e.g., Evans and Schmalensee (2010)).

That said, existing discussions on user-base externality are often static, leaving out inter-temporal effects that can be even more important. The fact that a larger user base today helps improve the technology tomorrow, and a larger anticipated user base tomorrow encourages greater investments today are examples of how user-base externality can play an inter-temporal role. Filecoin the data storage network, Dfinity the decentralized cloud service, marketplace such as overstock (and its ICO), and infrastructure projects such as Ethereum and LITEX all exhibit user-base externality in both contemporaneous and inter-temporal fashions, as our model highlight.

**Assets with Network Effect.** Given monetary embedding and user-base externality, tokens essentially constitute an asset that delivers owners "dividends" (in terms of user surplus) which increase in the scale of the platform. This type of assets are not restricted to tokens or blockchains. Network effect or user-base externality is prevalent and particularly

[16] According to CoinSchedule, 34.5% of the ICO-financed projects over the past two years focus on infrastructure. While the other top categories included trading and investing at 13.7%, finance at 10.2%, payments at 7.8%, data storage also at 7.8%, and drugs and healthcare at 5.5%, amongst dozens of other industry categories. Regardless the ICO category, these projects share user-base externality as a common attribute, and in terms of user adoption, exhibit a S-shaped development curve – the growth of user base feeds on itself.

important in the early stage of adoption for social networks and payment networks such as Facebook, Twitter, WeChat, and PayPal. Other examples of such assets include membership of clubs with benefits growing in the network size, and collectibles of limited edition for sports teams. Our model applies to such platforms or systems and their associated assets with network effect.

As such, even beyond blockchain-based decentralized application, the insights gleaned from our model helps inform what is likely to happen for other sharing economy applications such as UBER and AirBnB to introduce native currencies. In fact, consistent with our model, when Tencent QQ introduced Q-coin, a case to which our model is applicable, many users and merchants quickly started accepting them even outside the QQ platform (mapped to increase in system trade surplus in our model), tremendously accelerating adoption and token appreciation.[17]

Our model in the next Sections aims to capture these defining features of this class of assets with user base externalities of which cryptocurrencies and tokens are examples, and characterize the co-evolution of asset valuation and platform adoption.

# 3 The Model

## 3.1 Model setup

**Preferences and Off-blockchain Environment.** Consider a continuous-time economy with a unit mass of agents. Generic goods serve as numeraire ("dollar"). Agents' risk preference is given by a stochastic discount factor ("SDF") $\Lambda$ satisfying

$$\frac{d\Lambda_t}{\Lambda_t} = -rdt - \eta d\hat{Z}_t^{\Lambda}, \tag{1}$$

where $r$ is the risk-free rate and $\eta$ is the price of risk for systematic Brownian shock $\hat{Z}_t^{\Lambda}$ under the physical measure. Chen (2010) shows that the SDF in Equation (1) can be generated from a consumption-based asset pricing model.

Let $dZ_t^A$ denote the SDF shock under the risk-neutral measure. Using the Girsanov theorem, we have

$$dZ_t^{\Lambda} = d\hat{Z}^{\Lambda} + \eta dt. \tag{2}$$

[17]Annual trading volume reached billions of RMB in late 2000s and the government has to intervene. See articles *China bars use of virtual money for trading in real goods* and *QQ: China's New Coin of the Realm?* (WSJ).

Throughout this paper, we use "ˆ" indicate the physical measure. The SDF gives the price of Arrow-Debreu securities in each state of the world, representing off-blockchain investment opportunities.

**Blockchain Activities.** We model agents' investments in cryptocurrency or crypto-tokens (generically referred to as "token") directly under the risk-neutral measure. Later for model calibration and the discussion of cryto risk premium, we go back to the physical measure. The blockchain has a productivity $A_t$ that follows a geometric Brownian motion

$$\frac{dA_t}{A_t} = \mu^A dt + \sigma^A dZ_t^A, \tag{3}$$

where $\mu^A > 0$ and $\sigma^A > 0$. $A_t$ represents the quality or usefulness of blockchain platform. While pure technological shocks in cryptography or consensus algorithms obviously affect $A_t$, systematic shifts in user preferences, regulatory changes, and complementary innovations can all play a role.[18] A positive shock to $A_t$ captures an increase in the quality of technology or diversity of business activities feasible on the platform. In the benchmark model, we take $A_t$ process as exogenous, but later allow $A_t$ to depend on the user base.

Let $\rho$ denote the instantaneous correlation between the SDF shock and the blockchain productivity shock.[19] The usefulness of a particular platform evolves with the economy, as agents discover new ways to utilize the technology, which in turn depends on the progress of complementary technologies. As aforementioned, macro and regulatory events affect the usage of a blockchain platform. The crypto beta from $\rho$ is priced under the physical measure, generating a link between token price fluctuation and expected return.

A representative agent $i$ chooses the units of tokens carried forward, denoted by $k_{i,t}$, taking as given the current market price of token, $P_t$. We conjecture that the equilibrium

[18]For example, the effectiveness of the blockchain technology – provision of decentralized consensus – is affected by the protocol design and participants' behavior. Biais, Bisiere, Bouvard, and Casamatta (2017) study the stability of consensus, while Cong and He (2018) relate the quality of blockchain platform to miner/keeper activities. On the regulatory side, Bitcoin became popular in Greece during the country's financial distress in 2015. Interests and trading in Bitcoin rose quickly amidst fears of capital controls and the possibility of exiting the Eurozone (Lee and Martin (2018)). On the other extreme of regulatory impact, in 2017 and 2018, China and Korea have introduced various measures to limit cryptocurrency trading and usages, which are widely considered as a negative shock to cryptocurrencies that triggered widespread price declines. Athey, Parashkevov, Sarukkai, and Xia (2016) also consider a similar quality shock (probabilistic full breakdown) as a source of exogenous uncertainty in their model.

[19]Under the physical measure, $A_t$ has the following law of motion, $dA_t = \hat{\mu}^A A_t dt + \sigma^A A_t d\hat{Z}_t^A$, where using Girsanov theorem, we know that $\hat{\mu}^A$ is equal to $\mu^A + \eta\rho\sigma^A$, and $d\hat{Z}_t^A$ is the Brownian productivity shock under the physical measure, given by $d\hat{Z}_t^A = dZ_t^A - \eta\rho dt$.

dynamics of $P_t$ follows a diffusion process,

$$dP_t = P_t \mu_t^P dt + P_t \sigma_t^P dZ_t^A, \tag{4}$$

which agents take as given under rational expectation. We confirm this equilibrium conjecture once we clear the token market. Throughout this paper, we use capital letters for aggregate and price variables that individuals take as given, and lower-case letters for individual-level variables.

By holding $k_{i,t}$ units of token, agent $i$ obtains a flow of goods, which we interpret as certain types of *trade surplus* that can only be achieved on the blockchain by transacting in its native currency, i.e., tokens. The surplus can be a convenience yield in the case of transaction cryptocurrency (e.g., Bitcoin) or production flow from entrepreneurial projects in case of smart-contracting cryptocurrency (e.g., Ether).[20]Specifically, let $y_{i,t}$ denote the cumulative trade surplus at time $t$, then its flow over the next instant $dt$ is

$$dy_{i,t} = \left(P_t k_{i,t}\right)^{1-\beta} \left(N_t A_t e^{u_{i,t}}\right)^{\beta} dt, \tag{5}$$

which depends on the common productivity $A_t$, the idiosyncratic productivity $u_{i,t}$, and noticeably the user base $N_t$, that is the total number (measure) of agents that decide to join the blockchain network (i.e., $k_{i,t} > 0$, to hold some tokens). This specification captures strategic complementarity (or user-base externality) and reflects the ease to find trading or contracting counterparties in a large blockchain community.

We assume that to join the blockchain community and obtain this trade surplus, an agent has to incur a cost equal to $\chi dt$. This cost can be of the cognitive nature. For example, during a period of heightened media attention, it is easier to gather information about potential usages of the blockchain system and what needs to be prepared for a transaction on the blockchain. It can also represent efforts in trading on the blockchain, or writing smart contracts, or simply carrying out entrepreneurial activities on-chain. Importantly, agents may easily abstain from participating in the ecosystem any time and save the cost. This feature reflects the reality that in a decentralized blockchain-based system, switching between on-chain and off-chain is rather frictionless. When $\mu^A > 0$, i.e., the blockchain productivity grows over time, overwhelming the participation cost $\chi$, so everyone eventually

[20]Non-trivial transaction fees would naturally incentivize users to hold the native tokens; though not yet widely implemented, smart contracts may involve automated transfers for contingencies specified over an extended period of time, effectively requiring escrowing the tokens.

joins the community. We are interested in the initial adoption dynamics and its interaction with token price.

$u_{i,t}$ captures the heterogeneity in agents' needs to transact on the blockchain. Agents with low $u_{i,t}$ may not find it profitable to join the platform at time $t$ under this participation cost. The heterogeneity in $u_{i,t}$ makes the determination of user base non-trivial. Without such heterogeneity in the cross section, $N_t$ is either equal to zero (no one joins the community). To solve for the equilibrium $N_t$, we need to track the whole distribution of $u_{i,t}$. For simplicity, we assume that $u_{i,t}$ follows the Ornstein-Uhlenbeck process

$$du_{i,t} = -\mu^U u_{i,t}dt + \sigma^U dZ^U_{i,t}, \tag{6}$$

where $Z^U_{i,t}$ is an idiosyncratic, standard Brownian motion. The parameters $\mu^U$ and $\sigma^U$ are common among agents. And we set the initial cross-section distribution of $u_{i,0}$ equal to the stationary solution of the corresponding Fokker–Planck equation, i.e., with the following probability density

$$g(u) = \sqrt{\frac{1}{2\pi\theta^2}}e^{-\frac{u^2}{2\theta^2}} \text{ (where } \theta = \frac{\sigma^U}{\sqrt{2\mu^U}}\text{)}, \tag{7}$$

so that the cross-section distribution of $u_{i,t}$ does not vary over time. $\theta$ essentially describes the effective heterogeneity because it is the ratio of user heterogeneity scaled by the rate of mean reversion.

The interpretation of $u_{i,t}$ depends how we understand the blockchain trade surplus. For example, if we have in mind a payment blockchain (e.g., Bitcoin and Ripple), a high value of $u_{i,t}$ reflects agent $i$'s urge to conduct a transaction on the platform, be it an international remittance or a purchase of illegal drugs; if the trade surplus arises from smart contracting for business operations that are only possible on the blockchain (e.g., Ethereum), $u_{i,t}$ then reflects the productivity of such entrepreneurial projects; if the trade surplus derives from decentralized computation (e.g., Dfinity) or data storage (e.g., Filecoin), $u_{i,t}$ reflects the need for secure and fast access to computing power and data.

## 3.2 Dynamic Equilibrium

**Optimal Token Holdings.** An agent $i$ maximizes the following expected utility function under the risk-neutral measure

$$\mathbb{E}\left[\int_{t=0}^{\infty} e^{-rt}dc_{i,t}\right], \tag{8}$$

where $c_{i,t}$ is agent $i$'s *cumulative* consumption up to $t$. Let $w_{i,t}$ denote a representative agent $i$'s wealth at time $t$, which has the following law of motion

$$\begin{aligned} dw_{i,t} &= -dc_{i,t} + (w_{i,t} - P_t k_{i,t})\, r dt + \underbrace{(P_t k_{i,t})^{1-\beta} \left(N_t A_t e^{u_{i,t}}\right)^{\beta} dt}_{\text{blockchain trade surplus}} \\ &\quad + \underbrace{(P_t k_{i,t}) \left[\mu_t^P dt + \sigma_t^P dZ_t^A\right]}_{\text{capital gain of token holdings}} - \underbrace{\mathbb{I}_{\{k_{i,t}>0\}} \chi dt}_{\text{participation cost}} . \end{aligned} \tag{9}$$

Agent $i$ has three state variables, $w_{i,t}$, $u_{i,t}$, and $A_t$. We conjecture the value function takes the following form:

$$v_{i,t} = w_{i,t} + F\left(u_{i,t}, A_t\right). \tag{10}$$

Under this value-function conjecture, we can write the HJB equation of agent $i$ as follows.

$$\begin{aligned} rv_{i,t} &= rw_{i,t} + \max_{k_{i,t}} \left\{ (P_t k_{i,t})^{1-\beta} \left(N_t A_t e^{u_{i,t}}\right)^{\beta} - \chi + (P_t k_{i,t})\, \mu_t^P - P_t k_{i,t} r \right\} \mathbb{I}_{\{k_{i,t}>0\}} + \\ &\quad \frac{\partial F\left(u_{i,t}, A_t\right)}{\partial u_{i,t}} \left(-\mu^U u_{i,t}\right) + \frac{\partial F\left(u_{i,t}, A_t\right)}{\partial A_t} \mu_t^A A_t + \\ &\quad \frac{1}{2} \frac{\partial^2 F\left(u_{i,t}, A_t\right)}{\partial u_{i,t}^2} \left(\sigma^U\right)^2 + \frac{1}{2} \frac{\partial^2 F\left(u_{i,t}, A_t\right)}{\partial A_t^2} \left(\sigma_t^A\right)^2 A_t^2 + \frac{\partial F\left(u_{i,t}, A_t\right)}{\partial u_{i,t} \partial A_t} \sigma^U \sigma_t^A A_t. \end{aligned} \tag{11}$$

Note that consumption drops off from the HJB equation because the marginal value of wealth is equal to one: consuming one unit of goods adds one unit of flow utility but also decreases the value function by one through the reduction in wealth. Also, $w_{i,t} r$ shows up on both sides and cancels out. By the Feynman-Kac Theorem, the HJB equation then reveals that $F\left(u_{i,t}, A_t\right)$ is the present value of blockchain trade surplus. We confirm this conjecture of value function after we characterize more features of the Markov equilibrium.

The first-order condition is that if $k_{i,t} > 0$,

$$(1-\beta)\, P_t^{1-\beta} \left(\frac{N_t A_t e^{u_{i,t}}}{k_{i,t}}\right)^{\beta} + P_t \mu_t^P = P_t r. \tag{12}$$

Rearranging the equation, we have the following proposition

**Proposition 1.** *Agent i's optimal holding of tokens is given by*

$$k_{i,t} = \frac{N_t A_t e^{u_{i,t}}}{P_t} \left(\frac{1-\beta}{r - \mu_t^P}\right)^{\frac{1}{\beta}}. \tag{13}$$

*It has the following properties: (1) $k_{i,t}$ increases in $N_t$. (2) $k_{i,t}$ decreases in token price $P_t$.*

*(3) $k_{i,t}$ increases in $A_t$ and $u_{i,t}$. (4) $k_{i,t}$ increases in the expected token price change, $\mu_t^P$.*

Agents hold more token when the common productivity or their agent-specific transaction need is high, and also when the community is larger because it is easier to conduct trades in the ecosystem. Equation (13) also reflects the investment motive to hold tokens, that is $k_{i,t}$ increases in the expected token price appreciation, $\mu_t^P$.

**Endogenous User Base.** The maximized flow surplus from accessing the blockchain community can be solved under the optimal token holdings:

$$\begin{aligned}
&\max_{k_{i,t}} \left\{ (P_t k_{i,t})^{1-\beta} \left(N_t A_t e^{u_{i,t}}\right)^{\beta} dt - \chi dt + (P_t k_{i,t})\, \mu_t^P dt - P_t k_{i,t} r dt \right\} \mathbb{I}_{\{k_{i,t}>0\}} \\
&= \left\{ N_t A_t e^{u_{i,t}} \beta \left( \frac{1-\beta}{r-\mu_t^P} \right)^{\frac{1-\beta}{\beta}} - \chi \right\} \mathbb{I}_{\{k_{i,t}>0\}} dt
\end{aligned} \tag{14}$$

Apparently, if

$$N_t A_t e^{u_{i,t}} \beta \left( \frac{1-\beta}{r-\mu_t^P} \right)^{\frac{1-\beta}{\beta}} < \chi, \tag{15}$$

agent $i$ abstains from adoption and does not hold any tokens, i.e., $k_{i,t}$. Rearranging the equation, we can solve $\underline{u}_t$, the lower bound of $u_{i,t}$ for agents to adopt.

**Proposition 2.** *At time $t$, only agents with $u_{i,t} \geq \underline{u}_t$ adopt, where*

$$\underline{u}_t = \underline{u}\left(N_t; A_t, \mu_t^P\right), \tag{16}$$

*which increases in the user base $N_t$. Let $G(\cdot)$ denote the cumulative probability function of stationary cross-section distribution of $u_{i,t}$. The user base increases in $\mu_t^P$ and is given by*

$$N_t = 1 - G\left(\underline{u}_t\right). \tag{17}$$

We note that $N_t = 0$ is always an equilibrium. In such a case, blockchain trade surplus is zero. Under the risk-neutral measure, the expected token price change, $\mu_t^P$, is less than $r$, so when trade surplus is zero, agents will not participate. Also, $P_t = 0$ (and $N_t = 0$) is an equilibrium, because the dollar value of a token transfer is zero, so no trade can be done on the blockchain. We do not consider such degenerate equilibrium, and focus on the case where $N_t > 0$ and $P_t > 0$. We later show that in the space of $\mathbb{C}^1$, we have a unique Markov equilibrium.[21]

In Proposition 2, $N_t$ increases in $\mu_t^P$. This is a key economic mechanism that we emphasize. As $A_t$ grows, $N_t$ grows accordingly because more agents find it profitable to join the ecosystem by paying the cost $\chi$. In agents' expectation, a higher *future* $N$ implies a higher demand for tokens and thus a higher future price of token. As a result, the expectation of token price appreciation leads to higher $N_t$ *now*. The secondary market trading of tokens reflects expectation and induces a form of intertemporal complementarity in user base $N_t$.

Given the current common productivity $A_t$ and agents' expected price appreciation $\mu_t^P$, there may be multiple values of $N_t$ that satisfy Equations (**??**) and (**??**). To address the issue of multiplicity, consider the properties of a *response* function $R\left(n; A_t, \mu_t^P\right)$ that maps a hypothetical value of $N_t$, say $n$, to the measure of agents who choose to join the community after knowing $N_t$:

$$R\left(n; A_t, \mu_t^P\right) = 1 - G\left(\underline{u}\left(n; A_t, \mu_t^P\right)\right), \tag{18}$$

The equilibrium $N_t$ is the interaction between the $45^o$ line and the response curve as shown in the left panel of Figure 1. Note that given $A_t$ and $\mu_t^P$, $\underline{u}\left(n; A_t, \mu_t^P\right)$ approaches infinity, and $R\left(n; A_t, \mu_t^P\right)$ approaches zero, as $n$ approaches zero. Therefore, the response curve originates from zero. We have a unique equilibrium of positive $N_t$ if and only if the response curve crosses the $45^o$ line once in the range of $(0, 1]$, which is proved in the appendix. Figure 1 illustrates the equilibrium determination given $A_t$ and $\mu_t^P$, which we take as a snapshot of the dynamic equilibrium with time-varying productivity and expectation of price change.

**Lemma 1.** *Given $A_t$ and $\mu_t^P$, $N_t$ in Proposition 2 is unique because the response function only crosses the $45^o$ once.*

Note that the degenerate equilibrium of $N_t = 0$ can be ruled out by assuming a different functional form of trade surplus: $\left(P_t k_{i,t}\right)^{(1-\beta)} \left(A_t e^{u_{i,t}}\right)^{\beta} dt + \left(P_t k_{i,t}\right)^{(1-\beta)} N_t^{\beta} dt$, i.e., with $N_t$ entering the surplus in an additive form. Under this specification, there are always participating agents whose $u_{i,t}$ is high enough, which adds to $N_t$ and induce more agents to join the platform. The concavity in $N_t$ ensures a decreasing marginal impact of user-base expansion.

[21]Path continuity rules out the possibility that $P_t$ alternates between a positive value and zero over time (see a related discussion in Glasserman and Nouri (2016)), leaving us with only one non-degenerate Markov equilibrium of positive $P_t$ and one fully degenerate equilibrium where $P_t$ is zero at all $t$.

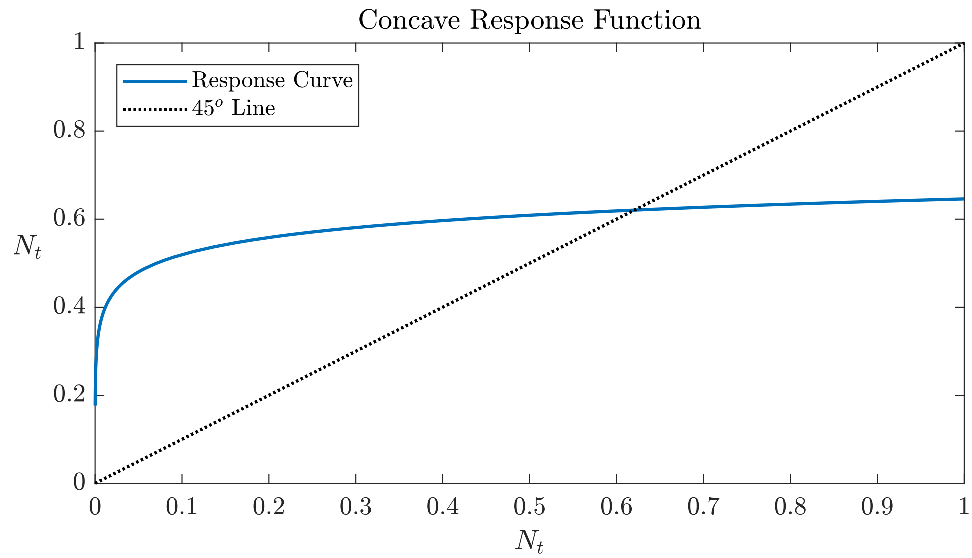


Figure 1: Determining User Base.

**Token Pricing Formula.** First, we define the *participants' average need for trade* as

$$S_t := \int_{\underline{u}_t}^{\infty} e^u g(u)\, du, \tag{19}$$

where $g(u)$ is the density function of $u_{i,t}$. $S_t$ is the average $e^{u_{i,t}}$ of participating agents. We consider a fixed supply of tokens, $\overline{M}$.[22] The market clearing condition is

$$\overline{M} = \int_{u_{i,t} \geq \underline{u}_t} k_{i,t} di, \tag{20}$$

Substituting in agents' optimal token holdings, we have the following proposition.

**Proposition 3.** *The token market clear condition offers a token price formula:*

$$P_t = \frac{N_t S_t A_t}{\overline{M}} \left( \frac{1-\beta}{r-\mu_t^P} \right)^{\frac{1}{\beta}}, \tag{21}$$

The token price increases in $N_t$, the size of blockchain user base – the larger the ecosystem is, the higher trade surplus individual participants can realize by holding tokens. Our model contribute to the literature of asset pricing by providing a theoretical foundation for the

[22]This is consistent with many ICOs that fix the supply of tokens. Because resources for business operations on-chain are all discussed in real terms, we can simply normalize $\overline{M}$ to one due to money neutrality.

commonly used valuation-to-user base ratio in the technology industry, especially popular for valuing firms whose customer base feed on endogenous network effects. The P-N ratio increases in the blockchain productivity, expected price appreciation, and the average agent-specific needs for trade, while decreases in the supply of tokens, $\overline{M}$. It worth emphasizing that the asset we price is a blockchain token, not equity stakes of firms. The formula also does not rely on exogenously given network size, and thus exhibits desirable features of valuation frameworks heuristically described in the industry, such as DAA (Daily Active Addresses) and NVT Ratio (market cap to daily transaction volume).[23]

**The Markov Equilibrium.** The token pricing formula suggests that there exists a Markov equilibrium with $A_t$ being the only aggregate state variable.

**Definition 1.** *For any initial value of $A_0$, the distribution of idiosyncratic productivity $u_{i,t}$ given by the density function $g(u)$, and any endowments of token holdings among the agents, $\{k_{i,0}, i \in [0,1]\}$, such that*

$$\overline{M} = \int_{i\in[0,1]} k_{i,0} di,$$

*a Markov equilibrium with state variable $A_t$ is described by the stochastic processes of agents' choices and token price on the filtered probability space generated by Brownian motion $\{Z_t^A, t \geq 0\}$ under the risk-neutral measure, such that*

*(1) Agents know and take as given the process of token price;*

*(2) Agents optimally choose consumption, and token and off-blockchain investments;*

*(3) Token price adjusts to clear the token market as in Proposition 3;*

*(4) All variables are functions of $A_t$ that follows an autonomous law of motion given by Equation (3) that maps any path of shocks $\{Z_s, s \geq t\}$ to the current state $A_t$.*

This conjecture of Markov equilibrium with state variable $A_t$ is consistent with the equilibrium conditions. For example, by Itô's lemma, $\mu_t^P$ is equal to $\left(\frac{dP_t/P_t}{dA_t/A_t}\right)\mu^A + \frac{1}{2}\frac{d^2P_t/P_t}{dA_t^2/A_t^2}\left(\sigma^A\right)^2$, which is a univariate function of $A_t$ in the Markov equilibrium. From Proposition 2, we can solve $N_t$, which only only depends on $A_t$ and $\mu_t^P$, and thus, is also a univariate function of $A_t$. Similarly, we can solve $\underline{u}_t$ as a function of $A_t$. These are consistent with Equation (21), the token pricing formula. Here, all the endogenous aggregate or variables, $P_t$, $\mu_t^P$, $N_t$, and $\underline{u}_t$ only depend on the single state variable $A_t$.

[23]See, for example, *Today's Crypto Asset Valuation Frameworks* by Ashley Lannquist at Blockchain at Berkeley and Haas FinTech.

In the Markov equilibrium, we confirm our value function conjecture. Substituting the optimal token holdings into the HJB equation, we have

$$\begin{aligned} rF\left(u_{i,t},A_t\right) = \quad & \max\left\{0, N_t A_t e^{u_{i,t}}\beta\left(\frac{1-\beta}{r-\mu_t^P}\right)^{\frac{1-\beta}{\beta}} - \chi\right\} + \frac{\partial F(u_{i,t},A_t)}{\partial u_{i,t}}\left(-\mu^U u_{i,t}\right) + \frac{\partial F(u_{i,t},A_t)}{\partial A_t}\mu^A A_t \\ & + \frac{1}{2}\frac{\partial^2 F(u_{i,t},A_t)}{\partial u_{i,t}^2}\left(\sigma^U\right)^2 + \frac{1}{2}\frac{\partial F(u_{i,t},A_t)}{\partial A_t}\left(\sigma^A\right)^2 A_t^2 + \frac{\partial F(u_{i,t},A_t)}{\partial u_{i,t}\partial A_t}\sigma^U\sigma^A A_t. \end{aligned} \tag{22}$$

On the right-hand side of the HJB equation, the first term is a function of $u_{i,t}$ and $A_t$, so overall, the HJB equation translates into a second-order differential equation for $F\left(u_{i,t},A_t\right)$.[24]

**Solving the Equilibrium.** So far, we have shown that once the token pricing function $P\left(A_t\right)$ is known, we can solve for $\mu_t^P$ using Itô's lemma, and then, the optimal token holdings, $k_{i,t}$ using Proposition 1. From Proposition 2, we solve for the user base, $N_t$, and the lower bound of participants' idiosyncratic productivity, $\underline{u}_t$. Substituting these variables into the token pricing formula (Equation (21)), we have the right-hand side depends only on $A_t$, $P\left(A_t\right)$, and the first and second derivatives of $P\left(A_t\right)$ in $\mu_t^P$, $N_t$, and $\underline{u}_t$. Therefore, the token pricing formula is a second-order ordinary differential equation ("ODE") for $P\left(A_t\right)$. By imposing proper boundary conditions, we can solve for $P\left(A_t\right)$.[25]

A unique solution of the second-order ODE requires boundary conditions. The first is

$$\lim_{A_t\to 0} P\left(A_t\right) = 0, \tag{23}$$

so when the platform is not productive any more, token price collapses to zero. For the second, we explore the asymptotic behavior of the economy. When $A_t$ approaches infinity, $N_t$ approaches one and $\underline{u}_t$ approaches $-\infty$, so even agents with extremely low transaction needs join the community. When $N_t = 1$, the participants' average need for trade is $S_t = e^{\theta^2/2}$. From the token pricing formula, we know the *asymptote* of $P\left(A_t\right)$ as $A_t$ grows to a very large

[24] A set of natural boundary conditions pin down a unique solution: $\lim_{A_t\to 0} F\left(u_{i,t},A_t\right) = 0$, $\lim_{A_t\to+\infty} F\left(u_{i,t},A_t\right) = +\infty$, $\lim_{u_{i,t}\to 0} F\left(u_{i,t},A_t\right) = 0$, $\lim_{u_{i,t}\to+\infty} F\left(u_{i,t},A_t\right) = +\infty$. Using the Feynman-Kac theorem, we can show that $F\left(u_{i,t},A_t\right) = \mathbb{E}_t\left[\int_{s=t}^{+\infty} e^{-r(s-t)}\max\left\{0, N_s A_s e^{u_{i,s}}\beta\left(\frac{1-\beta}{r-\mu_s^P}\right)^{\frac{1-\beta}{\beta}} - \chi\right\} ds\right]$.

[25] The existence of ODE requires a unique mapping from $A_t$, $P\left(A_t\right)$, and $P'\left(A_t\right)$ to $P''\left(A_t\right)$. The question is: given $A_t$ and $P_t$, can we uniquely solve $\mu_t^P$ using the token pricing formula? Since $\underline{u}_t$ decreases in $\mu_t^P$ (and $N_t$ increases in $\mu_t^P$), the right-hand side of the token market clearing condition increases in $\mu_t^P$, and we can uniquely pin down $\mu_t^P$ given $A_t$ and $P_t$.

value is

$$\overline{P}\left(A_{t}\right)=\frac{e^{\frac{\theta^{2}}{2}}A_{t}}{\overline{M}}\left(\frac{1-\beta}{r-\mu^{A}}\right)^{\frac{1}{\beta}},\tag{24}$$

where $N_t$ in the token pricing formula is replaced by 1 and the last integral is replaced by its limiting value $e^{\frac{1}{4\theta}}$. Moreover, note that $P_t$ is proportional to $A_t$, thus $\mu_t^P = \mu^A$ and we can use $\mu^A$ in Equation (24). To summarize, we have the following boundary condition.

**Lemma 2.** *There exists $\overline{A}$ such that*

$$P\left(\overline{A}\right)=\overline{P}\left(\overline{A}\right) \textit{ (value matching) and } P'\left(\overline{A}\right)=\overline{P}'\left(\overline{A}\right) \textit{ (smooth pasting).}\tag{25}$$

*For $A_t > \overline{A}$, $P(A_t) = \overline{P}(A_t)$ and $N_t = 1$.*

$\overline{A}$ is the upper bound of $A_t$, above which token price is equal to $\overline{P}(A_t)$ because $N_t = 1$.[26] We have exactly three boundary conditions for a second-order ODE and an endogenous boundary $\overline{A}$ that uniquely pin down the solution. Note that the equilibrium is unique in the space of $\mathbb{C}^1$, i.e., continuous and smooth functions, where we finds the ODE solution of $P(A_t)$. Outside $\mathbb{C}^1$, there can be multiple equilibria. Consider a positive token price at time $t$. Without restricting the price path to be continuous, at $t + dt$, agents can coordinate to $P_{t+dt} = 0$, and in the instant after this, agents can coordinate on a positive price again. The next proposition states the existence and uniqueness of Markov equilibrium.

**Proposition 4.** *There exists a unique Markov equilibrium with state variable $A_t$ that follows an autonomous law of motion given by Equation (3). Given the token price function $P(A_t)$, Proposition 2 solves the size of community $N_t$ and the lower bound of participant's idiosyncratic productivity $\underline{u}_t$ and Proposition 1 solves the optimal token holdings of participants $k_{i,t}$. Substituting these variables into the token pricing formula, i.e., Equation (21) in Proposition 3, we have a second-order ordinary differential equation that solves the token price function $P(\cdot)$ under the boundary conditions described in Equations (23) and (25).*

[26]Note that once the token price function $P(A_t)$ is solved, we know, by Itô's lemma, $\mu_t^P$ is a function of $A_t$, so Equation (22) gives a second-order partial differential equations for $F(u_{i,t}, A_t)$ with the boundary conditions discussed previously. However, to characterize the endogenous variables of interest, such as the community size $N_t$, we do not need to solve out $F(u_{i,t}, A_t)$.

# 4 Quantitative Analysis

## 4.1 Calibration

Our calibration is guided by the growth of token price and blockchain user base in the period from July 2010 and April 2018. In the model, since we fix the supply of tokens at $\overline{M}$, the variation of token price $P_t$ drives that of market capitalization (i.e., $P_t\overline{M}$). We map the dynamics of $P_t$ to that of the *aggregate* market capitalization of 16 major cryptocurrencies.[27] Since we study a representative token economy, we choose to focus on the aggregate token valuation that averages out idiosyncratic movements due to specificalities of token protocols and highlights the common feature, which is decentralized ledger or computing platform powered by the blockchain technology.

Accordingly, we collect the number of active user addresses for major cryptocurrencies, and map the aggregate number to $N_t$ in our model. Since the beginning month of our sample is unlikely to be the initial date of blockchain application to peer-to-peer platform, we choose to map the maximum number of active addresses, which was achieved in December 2017, to $N_t = 0.5$ in our model, and scale the number of active addresses in other months. As a result, we focus on the model performances in the states where $N_t \in [\underline{N}, 0.5]$, i.e., the early stage of adoption. $\underline{N}$ will be explained later together with Figure 3. We adjust parameters so that the model generates the following patterns in data: (1) the growth of $N_t$ over time; (2) the evolution of the growth rate of $N_t$; (3) the co-movement between $P_t$ and $N_t$. We later juxtapose our model-generated results along these dimensions together with the data.

The key parameters for the equilibrium dynamics of $N_t$ and $P_t$ are $\sigma^A$, $\beta$, $\theta$, and $\mu^A$. First, $\sigma^A$ and $\mu^A$ determines the time scale of this economy, i.e. how fast $N_t$ and $P_t$ grows, by pinning down the growth rate of $A_t$ *under the physical measure.* $A_t$ evolves as follows,

$$\frac{dA_t}{A_t} = \hat{\mu}^A dt + \hat{\sigma}^A d\hat{Z}_t, \tag{26}$$

where by the Girsanov theorem, $\hat{\mu}^A$ is equal to $\mu^A + \eta\rho\sigma^A$, and $\hat{Z}_t$ is a Brownian motion under the physical measure. $\eta$ is set to 1, roughly in line with the maximum Sharpe ratio given by the efficient frontier of U.S. stock market. $\rho$ is set to 1, a conservative value relative

[27] We include all cryptocurrencies with complete market cap and active address information on bitinfocharts.com, namely, AUR (Auroracoin), BCH (Bitcoin Cash), BLK (BlackCoin), BTC (Bitcoin), BTG (Bitcoin Gold), DASH (Dashcoin), DOGE (DOGEcoin), ETC (Ethereum Classic), ETH (Ethereum), FTC (Feathercoin), LTC (Litecoin), NMC (Namecoin), NVC (Novacoin), PPC (Peercoin), RDD (Reddcoin), VTC (Vertcoin). They represent more than 2/3 of the entire crypto market.

**Table 1: Calibration**

| | *Parameter* | *Value* | *Model* | *Data* |
|---|---|---|---|---|
| | | | Panel A: Key Parameters | |
| (1) | $\sigma^A$ | 200% | Growth rate of $N_t$ | Growth of user address |
| (2) | $\theta$ | $10/\sqrt{2}$ | Curvature of $N_t$ growth | Curvature of user-address curve |
| (3) | $\beta$ | 0.3 | Comovement: $N_t$ & $P_t$ | User address & crypto market cap |
| (4) | $\mu^A$ | 2% | $\frac{\sigma_t^N}{N_t}$, vol. of $N_t$ % change | Vol. of user-address % change |
| | | | Panel B: Other Parameters | |
| (5) | $\chi$ | 1 | Scaling effect on $A_t$ | |
| (6) | $\overline{M}$ | 1 billion | Monetary Neutrality | |
| (7) | $\rho$ | 1 | Shock correlation: SDF & $A_t$ | |
| (8) | $r$ | 5% | Risk-free rate | |
| (9) | $\eta$ | 1 | Price of risk | |

to the beta of technology sector (Pástor and Veronesi (2009)). We set one unit of time to be one year, and $r = 5\%$.

Because $P_t$ converges to a multiple of $A_t$, $\mu_t^P$ converges to $\mu^A$. Under the risk-neutral measure, the expected token price change has to be lower than $r$ (otherwise agents invest as much as they can in tokens), so we set $\mu^A$ equal to 2%. The gap between $\mu^A$ and $r$ determines how widely $\mu_t^P$ varies, so we use the volatility of percentage change of user addresses to discipline the choice of this parameter. Specifically, we map the data moment to average $\sigma_t^N/N_t$ in the states where $N_t \in [\underline{N}, 0.5]$.

Importantly, $\sigma^A$ is set to 200%, which contributes most to the growth rate of 202% under the physical measure, i.e., $\hat{\mu}^A$. As previously discussed, our interpretation of $A_t$ emphasizes the general usefulness of blockchain platform rather than narrowly defined technological progress. The type of activities on blockchain depends on the progress of complementary and competing technologies, government regulations, and critically, users' creativity and perception of the technology, suggesting a fast yet volatile growth of $A_t$. Given this annual growth rate of $A_t$, $N_t$ the sample span of approximately eight years implies the growth of $N_t$ from 0.0001 to 0.5, which corresponds precisely to the growth of normalized number of addresses in data.

While $\sigma^A$ pins down the growth rate of $N_t$, $\theta$ is responsible for the curvature of its growth path. In data, user-base growth rate rises over time, which will be shown to be qualitatively consistent with the model dynamics (i.e., a S-shaped development of $N_t$). To quantitatively match this pattern, we set $\theta = 10/\sqrt{2}$ (i.e., a cross-section variance of 50 for $u_{i,t}$).

Parameter $\beta$ governs the co-movement between $N_t$ and $P_t$. We can think of token as an asset that pays a flow dividend (i.e., the blockchain trade surplus). $\beta$ governs the decreasing return to $N_t$ in the dividend flow. Therefore, a higher value of $\beta$ increases the co-movement between $P_t$ and $N_t$ through a cash flow channel. We set $\beta$ to 0.3 so that the model generates the joint dynamics of $P_t$ and $N_t$ in the states where $N_t \in [0, 0.5]$.

The remaining parameters do not affect much the equilibrium dynamics. Recall that $\chi$ is the cost of joining the blockchain community, measured in goods. We set $\chi$ equal to 1 as a reference point for other parameters. We set $\overline{M}$, the supply of tokens, to 10 billion. Our model features monetary neutrality, that is the equilibrium outcome stays the same if $\overline{M}$ is doubled and $P_t$ halved.

## 4.2 Dynamic Adoption and Token Valuation

**Token price.** The token price is solved as a function of blockchain productivity $A_t$. The left panel of Figure 2 plots $dP_t/dA_t$ against $\ln(A_t)$. The curve starts at $A_t = 1e - 21$ (i.e., $\ln(A_t) = -48.35$), a number close to zero, the left boundary. The curve ends at the upper bound $\overline{A}$ where $P_t$ touches its asymptote, and as shown in the figure, its derivative equals that of $\overline{P}_t$. Over time, token price becomes increasingly sensitive to the variation in $A_t$. When the user base is small, token price is less responsive to the growth of $A_t$, because $A_t$ is multiplied by $N_t$ when entering into the blockchain trade surplus. As $A_t$ grows and $N_t$ approaches 1, $P_t$ becomes more sensitive to $A_t$.

The right panel of Figure 2 plots the logarithm of token price against the size of user base, both being functions of $A_t$ in the Markov equilibrium. This graph is particularly interesting because it links token price to various stages of adoption. On a logarithm scale, token price increases fast in the early stage, and then gradually rises with user base, but when the system has accumulated a critical mass of users, the increase of token price speeds up and converges to its long-run path. The comovement of $P_t$ and $N_t$ matches quite well the pattern in data, for which we only observe the early stage of adoption. Next, we will explore more in detail the dynamics of user adoption.

**User Base.** The solid line in Figure 3 plots the user base $N_t$ against the logarithm of $A_t$. The curve exhibits S-shaped development. When the blockchain technology is not so efficient, the growth of user base in response to technological progress is small. But as $N_t$ increases, the growth of user base feeds on itself – the more agents join the ecosystem, the higher surplus it is from trading on the blockchain. As a result, the growth of $N_t$ speeds up

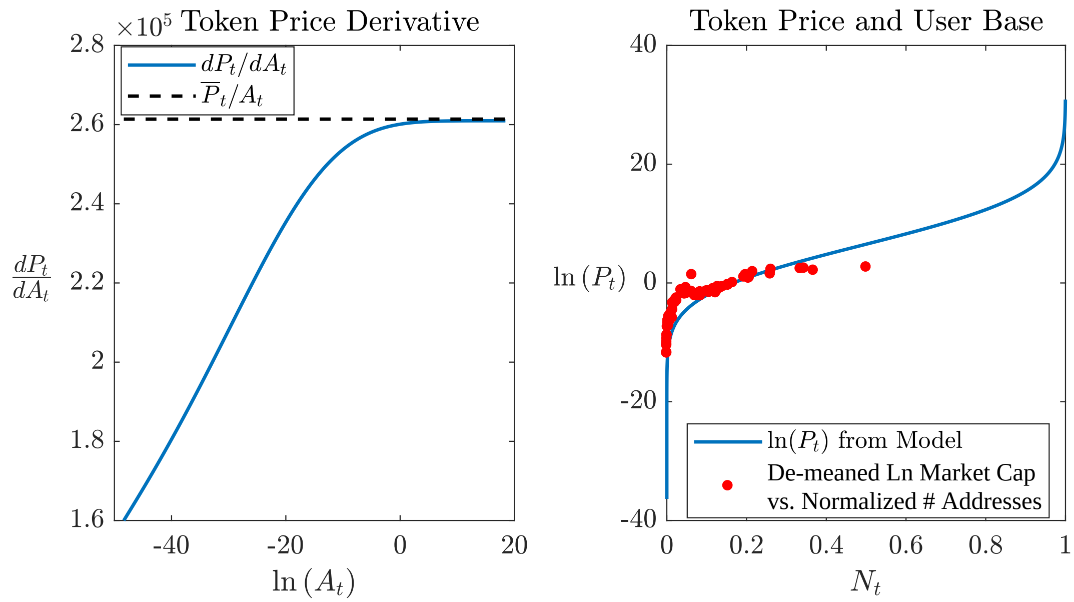


Figure 2: Dependence of Token Price on Blockchain Productivity and User Base.

in the interim range of blockchain productivity. User adoption eventually slows down when the pool of newcomers get exhausted. This model does not feature population growth, on the basis that population growth relative to changes in $A_t$ is small. This could be relaxed as long as when more agents join the ecosystem, there are less new comers left to be included if $A_t$ rises up further in the future.

Both the growth rate and curvature of $N_t$ over time match well with the pattern in data. As previously discussed, we map the highest number of user addresses (December 2017) to $N_t = 0.5$, and record the corresponding value of $\ln(A_t)$ in our model. We scale the number of addresses in other months by that of December 2017. With December 2017 as the reference point, we calculate the corresponding value of $\ln(A_t)$ for each observation by applying the annualized growth rate of 202% to the value of $\ln(A_t)$ in December 2017. The leftmost state has $N_t = \underline{N} = 0.0008$.

Figure 3 also compares the user adoption with and without tokens or blockchain native tokens. The former strictly dominates the latter. The two eventually converge to one as $A_t$ grows. Why introducing tokens speeds up user adoption? Next, we considers an ecosystem without tokens, and solve agents' participation decisions.

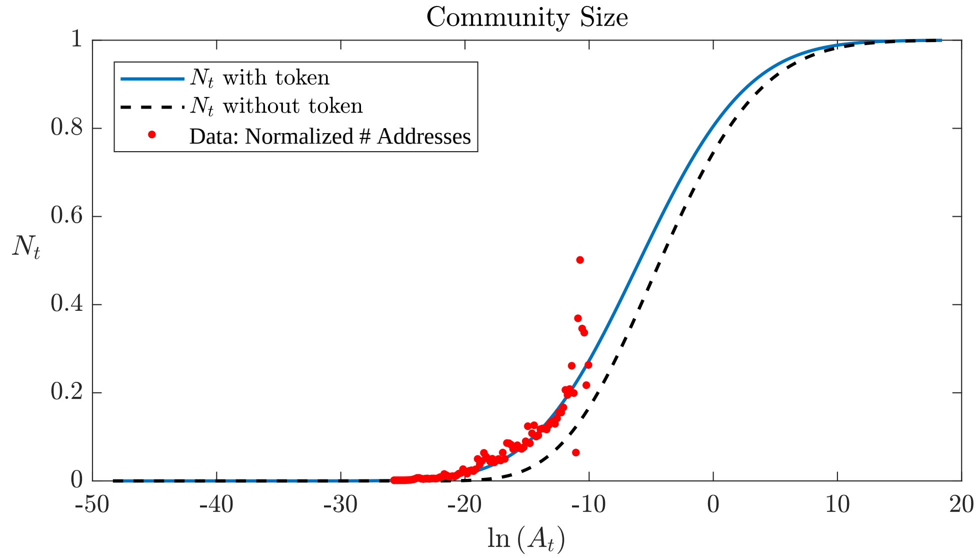


Figure 3: Dependence of User Base on Blockchain Productivity

### 4.3 The Growth and Volatility Effects of Token

When token is introduced as the required means of payment on a platform, its market price reflects agents' anticipation of future technological progress and user adoption, which translates into the expected token price appreciation. Tokens therefore accelerates adoption because agents join the community not only to enjoy the trade surplus but also the return from rising token price.

To understand this effect thoroughly, let us consider an alternative setup where agents can conduct businesses and enjoy the trade surplus on blockchain without holding tokens. Let $b_{i,t}$ denote the dollar value of investment or purchase done on the blockchain. Under the assumption of risk neutrality, the cost of funds is $r$, so agents solve the following profit maximization problem

$$\max_{b_{i,t}} (b_{i,t})^{1-\beta} \left(N_t A_t e^{u_{i,t}}\right)^{\beta} - \chi - b_{i,t} r \tag{27}$$

The first order condition is

$$(1-\beta)\left(\frac{N_t A_t e^{u_{i,t}}}{b_{i,t}}\right)^{\beta} - r = 0. \tag{28}$$

Rearranging the equation, we have

$$b_{i,t} = N_t A_t e^{u_{i,t}} \left(\frac{1-\beta}{r}\right)^{\frac{1}{\beta}}. \tag{29}$$

Substituting the optimal $b_{i,t}$ into the profit function, we have the profit for $k_{i,t} > 0$ equal to

$$N_t A_t e^{u_{i,t}} \beta \left( \frac{1-\beta}{r} \right)^{\frac{1-\beta}{\beta}} - \chi. \tag{30}$$

**Proposition 5.** *In an economy without tokens, only agents with $u_{i,t} \geq \underline{u}_t^{NT}$ ("NT" for no token) adopt, where $\underline{u}_t^{NT}$ increases in the user base $N_t$ and is given by*

$$\underline{u}_t^{NT} = -\ln(N_t) + \ln\left(\frac{\chi}{A_t \beta}\right) - \left(\frac{1-\beta}{\beta}\right) \ln\left(\frac{1-\beta}{r}\right). \tag{31}$$

*Let $G(\cdot)$ denote the cumulative probability function of cross-section distribution of idiosyncratic productivity. The user base is*

$$N_t = 1 - G\left(\underline{u}_t^{NT}\right). \tag{32}$$

**The Growth Effect and Inter-temporal Feedback.** In comparison with Proposition 2, the only difference in this proposition is that without tokens, the price appreciation term, i.e., $\mu_t^P$, drops out in the lower bound of idiosyncratic productivity. Therefore, in states where $\mu_t^P > 0$, a blockchain system with tokens has a larger community. The intuition is simply that agents hold tokens to enjoy not only the trade surplus uniquely available on-chain (utility purchase), but also the token price appreciation (investment purchase). In a system without tokens, the investment-driven demand is shut down.

**Corollary 1.** *In states where $\mu_t^P > 0$ ($< 0$), a blockchain with tokens has a larger (smaller) user base than the one without.*

Intuitively, if $A_t$ is expected to grow fast (for example, due to a larger and positive $\mu^A$), $\mu_t^P$ tends to be positive. Therefore, by introducing a native token that is required to be means of payment, a blockchain system can capitalize future productivity growth in token price, and thereby accelerate adoption. This is the growth effect of introducing tokens in promising platforms (large and positive $\mu^A$).

In essence, token enables a feedback loop reflecting the inter-temporal complementarity of user base. By capitalizing future productivity growth and popularity (i.e., large $N$), token induces faster adoption in the early stage. As the user base is *expected* to expand fast, token price is expected to appreciate because the flow surplus is expected to be higher for all participants and the demand for tokens expected to be higher. The expected price appreciation feeds into a stronger investment-driven demand for tokens.

We note that a predetermined token supply schedule is important. If token supply can arbitrarily increase ex post, then the expected token price appreciation is delinked from the technological progress. Predeterminancy or commitment can only be credibly achieved through the decentralized consensus mechanism empowered by the blockchain technology. In contrast, traditional monetary policy has commitment problem – monetary authority cannot commit not to supply more money when its currency value is relatively high.

**The Volatility Reduction Effect of Token.** Without native tokens, agents' decision to participate is purely driven by the current level of blockchain productivity $A_t$ and their idiosyncratic transaction needs $u_{i,t}$. Therefore, under the stationary distribution of $u_{i,t}$, the user base $N_t$ varies only with $A_t$, and its volatility is tied to the exogenous volatility of blockchain productivity. Introducing tokens also changes the volatility of $N_t$ through the fluctuation of the expected price change, because now, agents' decision to participate also depends on $\mu_t^P$. Next, we show that token *reduces* the volatility of user base.

To derive the dynamics of $N_t$, we first conjecture that $N_t$ follows a diffusion process in equilibrium

$$dN_t = \mu_t^N dt + \sigma_t^N dZ_t^A. \tag{33}$$

Strictly speaking, $N_t$ follows a reflected (or "regulated") diffusion process that is bounded below at zero and bounded above at one, so we study the interior behavior of $N_t$. In the appendix, we solve the volatility of $N_t$ for the case without token and the one with token. The following proposition summarizes the results.

**Proposition 6.** *In an economy without tokens, the diffusion of $dN_t$ is*

$$\sigma_t^N = \left(\frac{g\left(\underline{u}_t^{NT}\right)}{1 - g\left(\underline{u}_t^{NT}\right)/N_t}\right)\sigma^A. \tag{34}$$

*In an economy with tokens, the diffusion of $dN_t$ is*

$$\sigma_t^N = \left(\frac{g\left(\underline{u}_t\right)}{1 - g\left(\underline{u}_t\right)/N_t}\right)\left[\sigma^A + \left(\frac{1-\beta}{\beta}\right)\left(\frac{\sigma_t^{\mu^P}}{r - \mu_t^P}\right)\right], \tag{35}$$

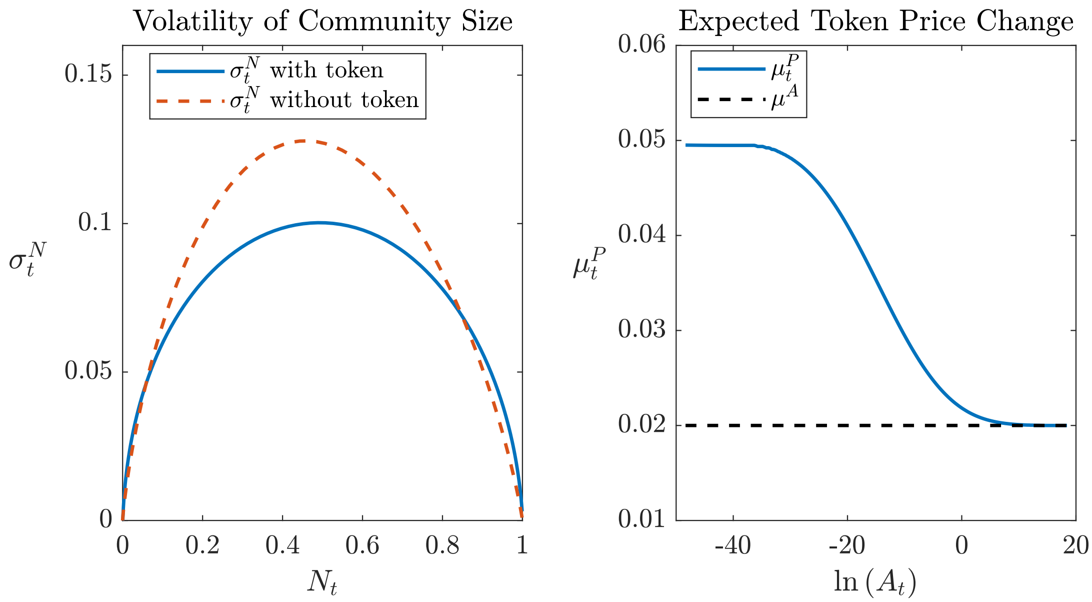


Figure 4: Volatility of User Base.

*where, $\sigma_t^{\mu^P}$ is the diffusion of $\mu_t^P$ in its law of motion,*

$$d\mu_t^P = \mu_t^{\mu^P} dt + \sigma_t^{\mu^P} dZ_t^A. \tag{36}$$

Comparing Equations (34) and (35), we see that introducing tokens alters the volatility dynamics of user base through the fluctuation of expected token price change, i.e., $\sigma_t^{\mu^P}$. A priori, having a native token may either amplify or dampen the shock effect on the the user base, depending on the sign of $\sigma_t^{\mu^P}$. By Itô lemma, $\sigma_t^{\mu^P} = \frac{d\mu_t^P}{dA_t}\sigma^A A_t$, so the sign of $\sigma_t^{\mu^P}$ depends on whether $\mu_t^P$ increases or decreases in $A_t$.

We note that $\mu_t^P$ decreases in $A_t$ (and thus, $\sigma_t^{\mu^P} < 0$), precisely because of the endogenous user adoption. Consider an increase in $A_t$, $N_t$ increases accordingly. However, in the future there are less newcomers remaining to join the community. Recall that token price appreciation is driven by the future increase in both $A_t$ and $N_t$, so when the potential for $N_t$ growth declines, the expected token price appreciation, i.e., $\mu_t^P$, declines.

The left panel of Figure 4 plots $\sigma_t^N$, and compares the cases with and without token across different stages of adoption. Apparently, introducing token reduces the volatility of $N_t$. Both curves starts at zero and ends at zero, consistent with the S-shaped development in Figure 3 where both curves starts flat and ends flat. This volatility reduction effect is

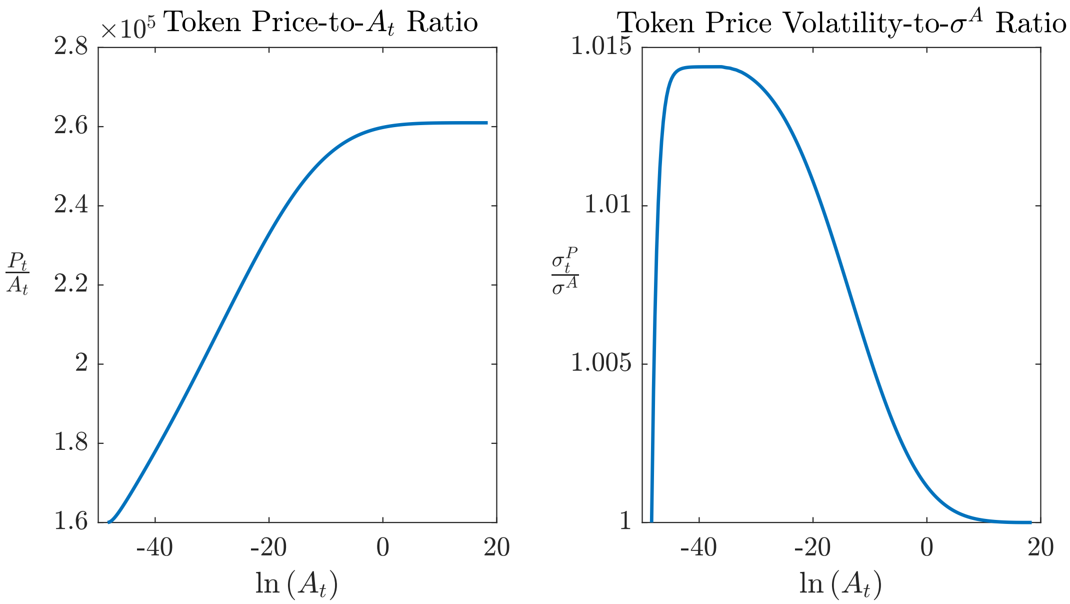


Figure 5: Token Price-to-Blockchain Productivity Ratio and Volatility.

more prominent in the early stage of development when $A_t$ and $N_t$ are low. Note that $\sigma_t^N$ can be slightly higher when token is introduced because the first brackets in Equations (34) and (35) can differ due to the difference between $\underline{u}_t^{NT}$ and $\underline{u}_t$ even for the same value of $N_t$. The right panel of Figure 4 plots $\mu_t^P$ against $\ln(A_t)$, showing their negative relation that causes $\sigma_t^{\mu^P} < 0$.

**The Run-up of Token Price Volatility.** The dynamics of user adoption in turn affects the volatility of token price. When $\chi = 0$, agents' decision to participate becomes irrelevant. Every agent participates, so $N_t = 1$, and token price is given by Equation (24) and the ratio of $P_t$ to $A_t$ is a constant. Therefore, the diffusion of token price, i.e., $\sigma_t^P$, is equal to $\sigma^A$.

A key theme of this paper is the endogenous dynamics of user adoption. When $\chi > 0$, the ratio of $P_t$ to $A_t$ depends on the $\underline{u}_t$, the threshold value of agent-specific needs for blockchain transactions, above which an agent participates. The variation of $N_t$ feeds into $P_t/A_t$, and therefore, amplifies the volatility of token price beyond $\sigma^A$, the level of volatility when the issue of user adoption is irrelevant.

The left panel of Figure 5 plots $P_t/A_t$. As shown in Equation (21), this ratio follows closely the dynamics of $N_t$ shown in Figure 3, but is steeper in the early stage (i.e., low $A_t$ region). The right panel of Figure 5 plots the ratio of token price volatility to $\sigma^A$, which eventually converges to 1 as $N_t$ approaches one and $P_t$ approaches its asymptote. At its height, endogenous user adoption amplifies token price volatility (or instantaneous standard deviation, to be precise) by 1.4%. What is interesting is the qualitative implications: as a new platform is gradually adopted, one may observe dormant token price variation, followed by a volatility run-up before the eventual stabilization. A key result of this paper is this the mutual effect between $N_t$ and $P_t$ in both growth and volatility. Especially in the early stage of adoption, user participation amplifies the growth and volatility of token price, while at the same time, is being affected by agents' expectation of future token price appreciation and the volatility of such expectation.

### 4.4 Risk and Return under the Physical Measure

The expected token price appreciation under the physical measure is

$$\hat{\mu}_t^P = \mu_t^P + \eta\rho\sigma_t^P. \tag{37}$$

The covariance between token price change and SDF shock, i.e., $\rho\sigma_t^P$, is priced at $\eta$. If the shock to blockchain productivity is orthogonal to SDF shock ($\rho = 0$), then $\hat{\mu}_t^P = \mu_t^P$. Next, we graphically illustrate how risk premium varies across different stages of platform development. Figure 6 plots the token risk premium, i.e., $\hat{\mu}_t^P - r$, under the physical measure against $\ln(A_t)$. As $A_t$ increases, the risk-neutral drift of token price ($\mu_t^P$) declines, while the volatility ($\sigma_t^P$) follows a hump-shaped curve, which dominates the dynamics of risk premium.

Our risk premium of 200% is higher that the average annual return to the cryptocurrency portfolio in our sample (27%). The main reason is the decline of cryptocurrency market value in 2018. In the next section, we discuss an extension of the model that features time-varying token beta, and therefore, may generate such a decline and a risk premium more in line with data.

## 5 Discussion and Extension

### 5.1 Endogenous Growth: from User Base to Productivity

In this paper, our primary focus is on the endogenous and joint dynamics of token price and user base. Through token price that reflects agents expectation, the popularity of a platform in the future increases the present user base. Our analysis thus far has taken the blockchain productivity process as exogenous. In reality, many token and cryptocurrency

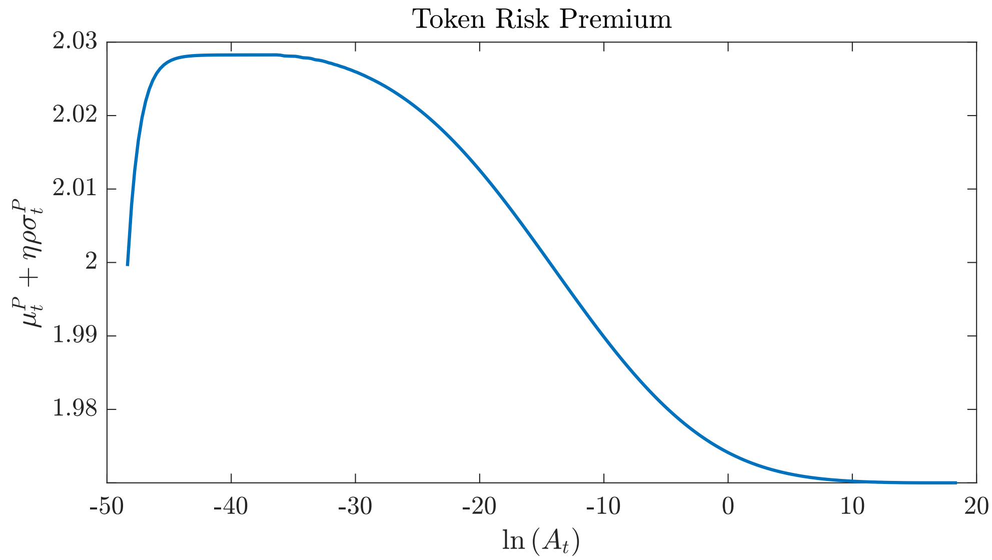


Figure 6: Expected Capital Gain under the Physical Measure.

applications feature an endogenous response of platform productivity to the variation of user base.

A defining feature of blockchain technology is the provision of consensus on decentralized ledgers. In a "proof-of-stake" system, the consensus is more robust when the user base is large and dispersed because no single party is likely to hold a majority of stake; in a "proof-of-work" system, more miners potentially deliver faster and more reliable confirmation of transactions, and miners' participation in turn depends on the size of user base through the associated media coverage (attention in general), transaction fees, and token price. More broadly, $A_t$ represents the general usefulness of the platform. When more users participate, more types of activities can be done on the blockchain. Moreover, a greater user base potentially directs greater resources and research into the blockchain community, accelerating the technological progress.

The endogeneity of blockchain productivity and its dependence on the user base highlight the decentralized nature of this new technology. To reflect this fact and discuss its theoretical implications related to the growth and volatility amplification effects, we modify the process of $A_t$ as follows:

$$\frac{dA_t}{A_t} = \left(\mu_0^A + \mu_1^A N_t\right) dt + \sigma^A dZ_t. \tag{38}$$

By inspection of Equation (5), the definition of trade surplus, it seems that $A_t$ and $N_t$ are not separately identified from the perspective of individual users, because either of the two is simply part of the marginal productivity. However, this argument ignores the fact that by feeding $N_t$ into the process of $A_t$, the growth rate of $A_t$ is no longer i.i.d.

Consider the case where $d\mu^A(N_t)/dN_t > 0$. A higher level of $N_t$ now induces faster growth of $A_t$, which leads to a higher level of $N_t$ in the future. Similarly, a lower current level of $N_t$ translates into a downward shifting of the path of $N_t$ going forward. In other words, the endogenous growth of $A_t$ induces *persistence* in $N_t$. In our benchmark setting, $N_t$ is reset every instant, depending on the exogenous level of $A_t$. Yet, here path dependence arises, which tends to amplify both the growth and unconditional volatility of $N_t$ by accumulating and propagating shocks to $A_t$. A formal analysis of this extension is certainly important in the light of improving quantitative performances of the model.

Another way to achieve such path dependence is to assume that agents' decision to join the community or quit incurs an adjustment cost, so $N_t$ becomes the other aggregate state variable, just as in macroeconomic models where capital stock becomes an aggregate state variable when investment is subject to adjustment cost. However, this specification does not capture the endogenous growth of $A_t$.

## 5.2 New Economy, Token Beta, and "Bubble"

So far, we have fixed the correlation between SDF shock and shock to $A_t$ as a constant. Yet as a blockchain platform or the general technology gains popularity, eases, its token is becoming a systematic asset. Pástor and Veronesi (2009) emphasize that the beta of new technology tends to increases as it becomes mainstream and well adopted. Here, we allow the correlation between SDF and $A_t$ to depend on $N_t$, and study its implications on token price. Specifically, we introduce a $N_t$-dependent volatility of $A_t$ shock such that $d\sigma_t^A/dN_t > 0$.

Under the physical measure, we consider $A_t$ following a geometric Brownian motion:

$$\frac{dA_t}{A_t} = \hat{\mu}^A dt + \sigma^A(N_t)\, d\hat{Z}_t, \tag{39}$$

where $\hat{\mu}^A$ and $\sigma^A$ are positive constants. Under the risk-neutral measure, we have

$$\frac{dA_t}{A_t} = \left[\hat{\mu}^A - \eta\rho\sigma^A(N_t)\right] dt + \sigma^A(N_t)\, dZ_t. \tag{40}$$

Once we have the dynamics of $A_t$ under the risk-neutral measure, we can proceed to solve the model with only one modification, that is the expected growth rate of $A_t$ is $\hat{\mu}^A - \eta\rho\sigma^A(N_t)$, which now declines in $N_t$.

Therefore, as $A_t$ grows, there are two opposing forces that drive $P_t$. On the one hand, the mechanisms that increase $P_t$ are still there: when $A_t$ directly increases the flow utility of token, or indirectly through $N_t$, token price increases. On the other hand, through the increase of $N_t$, the expected growth of $A_t$ under the *risk-neutral* measure declines, which pushes $P_t$ down. If our previous mechanisms work in the early stage of adoption while the channel of $N_t$-dependent token beta dominates in the later stage of adoption, what we shall see in the equilibrium will be a bubble-like behavior – $P_t$ rises initially, and later as $N_t$ rises, $P_t$ declines because the risk-neutral expectation of $A_t$ growth declines. Admittedly, this depends on the specific functional form that links $N_t$ to the volatility of $A_t$. The learning mechanism in Pástor and Veronesi (2009) may serve as a microfoundation, and there could be other channels that the user base affects the volatility of productivity shock.

## 5.3 Alternative Tokens and Boundary Conditions

Many blockchain platforms accommodate not only their native tokens but also other cryptocurrencies. For example, any ERC-20 compatible cryptocurrencies are accepted on Ethereum.[28] To address this issue, we may consider an alternative upper boundary of $A_t$. Define $\psi$ as the cost of creating a new cryptocurrency that is perfect substitute with the token we study because it functions on the same blockchain and therefore faces the same common blockchain productivity and agent-specific trade needs. This creates a reflecting boundary at $\overline{A}$ characterized by a value-matching condition and a smooth-pasting condition:

$$P\left(\overline{A}\right) = \psi \text{ and } P'\left(\overline{A}\right) = 0. \tag{41}$$

When token price increases to $\psi$, entrepreneurs outside of the model will develop a new cryptocurrency that is compatible with the rules of our blockchain system. So, the price level never increases beyond this value. Because it is a reflecting boundary, we need to rule out jumps of token prices, so the first derivative of $P\left(A_t\right)$ must be zero. Again we have exactly three boundary conditions for a second-order ODE and an endogenous upper boundary that uniquely pins down the solution.

Similarly, we may consider potential competing blockchain systems, and interpret $\psi$ as the cost of creating a new blockchain system and its native token, which together constitute a perfect substitute for our current system. This creates the same reflecting boundary for token price. When token price increases to $\psi$, entrepreneurs outside of the model will build

[28] ERC-20 defines a common list of rules for all tokens or cryptocurrencies should follow on the Ethereum blockchain.

a new system.

**Proposition 7.** *The upper boundary condition is given by Equation (41) in the two following cases: (1) the blockchain system accepts alternative tokens or cryptocurrencies that can be developed at a unit cost of $\psi$; (2) an alternative blockchain system that is a perfect substitute of the current system can be developed at a cost of $\psi$ per unit of its native tokens.*

While our framework accommodates the effect of competition, a careful analysis of crypto industrial organization certainly requires more ingredients, especially those that can distinguish between the entry of multiple cryptocurrencies into one blockchain system and competing blockchain systems.

## 5.4 Token Supply Schedule

In practice, many cryptocurrencies and tokens feature an increasing supply over time (for example, Bitcoin) or state-contingent supply in order to stabilize token price (for example, Basecoin). Our framework can be modified to accommodate this feature, and thus, serve as a platform for experimenting the impact of token supply on user base growth and token price stability. For example, we may consider the law of motion of token supply $\overline{M}$ given by an exogenous stochastic process, for example, as follows

$$d\overline{M}_t = \mu^M \left(\overline{M}_t, N_t\right) dt + \sigma^M \left(\overline{M}_t, N_t\right) dZ_t^A. \tag{42}$$

A new Markov equilibrium features two aggregate state variable, $A_t$ and $\overline{M}_t$.

An alternative formulation is to consider Poisson-arriving incremental of $\overline{M}$, which closely resemble the Bitcoin supply schedule. This formulation has the analytical advantage that equilibria between two Poisson arrivals still have only one state variable $A_t$. We can solve the model in a backward induction fashion, starting from the asymptotic future where token supply has plateaued and moving back sequentially in the Poisson time given the value function from the previous step.

As in many macroeconomic models, our framework features monetary neutrality: doubling the token supply *from now on* simply reduces token price by half and does not impact any real variables. However, neutrality is only achieved if the change of token supply is implemented uniformly and proportionally for any time going forward. If token supply is adjusted on a contingent basis, agents' expectation of token price appreciation will be affected, through which supply adjustments influence user base, token demand, and the total trade surplus realized on the platform.

Finally, we emphasize that to achieve the desirable effects of a token supply schedule, the schedule must be implemented automatically without centralized third-party interventions, so that dispersed agents take the supply process as given when making decisions. Such commitment to rules and protocols highlights a key difference between cryptocurrency supply and money supply by governments – through the discipline of decentralized consensus, blockchain developers can commit to a token supply schedule. A clearly defined mandate for central bankers (e.g., inflation and employment targeting) reflects the push for commitment to a state-contingent monetary policy, but in reality, discretions abound, especially in the face of unforeseeable events and political regime transitions.

# 6 Conclusion

This paper provides the first dynamic pricing model of cryptocurrencies and tokens, taking into consideration their role as required means of payment and user-base externalities under endogenous adoption. Our model highlights a key benefit of introducing cryptocurrencies and crypto-tokens on blockchain platforms: when agents expect future technology or productivity progress, token price appreciation induces more agents to join the platform by serving as an attractive investment. In other words, token capitalizes future growth and speeds up user adoption. We characterize the inter-temporal feedback mechanism and show it leads to an S-shaped adoption curve in equilibrium: dormant activities precede explosive growth before eventual stabilization. The growth effect of tokens enhances welfare and tokens can reduce the volatility of user adoption through price variation. Moreover, token price dynamics crucially depend on the platform productivity, endogenous user adoption, and user heterogeneity. More generally, our framework can be applied to dynamic pricing of assets with user-base externalities on a platform or in an ecosystem.

# Appendix I - Proofs

## AI.1 Proof of Proposition 1

The first derivative of $R\left(n; A_t, \mu_t^P\right)$ is

$$\frac{dR\left(n; A_t, \mu_t^P\right)}{dn} = -\left(\sqrt{\frac{\theta}{\pi}}e^{-\theta\underline{u}^2}\right)\frac{d\underline{u}}{dn} = \left(\sqrt{\frac{\theta}{\pi}}e^{-\theta\underline{u}^2}\right)\frac{1}{n} \tag{43}$$

Next, we solve the second derivative

$$\begin{aligned}\frac{d^2R\left(n; A_t, \mu_t^P\right)}{dn^2} &= -\left(\sqrt{\frac{\theta}{\pi}}e^{-\theta\underline{u}^2}\right)\frac{2\theta\underline{u}}{n^2} - \left(\sqrt{\frac{\theta}{\pi}}e^{-\theta\underline{u}^2}\right)\frac{1}{n^2} \\ &= -\left(\sqrt{\frac{\theta}{\pi}}e^{-\theta\underline{u}^2}\right)\frac{(2\theta\underline{u}+1)}{n^2}. \end{aligned}\tag{44}$$

Note that in the expression $\underline{u}$ is pinned down by $n$, and it is a decreasing function of $n$. When $n$ is small, $\underline{u}$ is large, so the response function starts from the origin as a concave curve, and a potential second crossing with the $45^o$ line happens when the response function turns convex and crosses from below as $n$ increases. For the equilibrium $N_t$ to be unique, a necessary and sufficient condition is that the second crossing never happens, i.e., $R\left(1; A_t, \mu_t^P\right) < 1$, i.e., $G\left(\underline{u}\left(1; A_t, \mu_t^P\right)\right) > 0$, or equivalently, $\underline{u}\left(1; A_t, \mu_t^P\right)$ is not equal to negative infinity, which trivially holds as shown in Equation (**??**).

**Proof of Proposition 6**. First, we consider the case without token. Using Itô's lemma, we can differentiate Equation (32), and then, by matching coefficients with Equation (33), we can derive the expressions for $\mu_t^N$ and $\sigma_t^N$:

$$dN_t = -g\left(\underline{u}_t^{NT}\right)d\underline{u}_t^{NT} - \frac{1}{2}g'\left(\underline{u}_t^{NT}\right)\left\langle d\underline{u}_t^{NT}, d\underline{u}_t^{NT}\right\rangle, \tag{45}$$

where $\left\langle d\underline{u}_t^{NT}, d\underline{u}_t^{NT}\right\rangle$ is the quadratic variation of $d\underline{u}_t^{NT}$. Using Itô's lemma, we differentiate Equation (31)

$$\begin{aligned} d\underline{u}_t^{NT} &= -\frac{1}{N_t}dN_t + \frac{1}{2N_t^2}\left\langle dN_t, dN_t\right\rangle - \frac{1}{A_t}dA_t + \frac{1}{2A_t^2}\left\langle dA_t, dA_t\right\rangle \\ &= -\left(\frac{\mu_t^N}{N_t} - \frac{\left(\sigma_t^N\right)^2}{2N_t^2} + \mu^A - \frac{\left(\sigma^A\right)^2}{2}\right)dt - \left(\frac{\sigma_t^N}{N_t} + \sigma^A\right)dZ_t^A \end{aligned}\tag{46}$$

Substituting this dynamics into Equation (45), we have

$$
\begin{aligned}
dN_t = & \left[ g\left(\underline{u}_t^{NT}\right)\left(\frac{\mu_t^N}{N_t} - \frac{\left(\sigma_t^N\right)^2}{2N_t^2} + \mu^A - \frac{\left(\sigma^A\right)^2}{2}\right) - \frac{1}{2}g'\left(\underline{u}_t^{NT}\right)\left(\frac{\sigma_t^N}{N_t} + \sigma^A\right)^2\right] dt \\
& + g\left(\underline{u}_t^{NT}\right)\left(\frac{\sigma_t^N}{N_t} + \sigma^A\right) dZ_t^A, \qquad (47)
\end{aligned}
$$

By matching coefficients on $dZ_t^A$ with Equation (33), we can solve $\sigma_t^N$.

Next, we consider the case with token. Once token is introduced, $N_t$ depends on the expected token price appreciation $\mu_t^P$. which is also a univariate function of state variable $A_t$ because by Itô's lemma, $\mu_t^P$ is equal to $\left(\frac{dP_t/P_t}{dA_t/A_t}\right)\mu^A + \frac{1}{2}\frac{d^2P_t/P_t}{dA_t^2/A_t^2}\left(\sigma^A\right)^2$. In equilibrium, its law of motion is given by a diffusion process

$$
d\mu_t^P = \mu_t^{\mu^P} dt + \sigma_t^{\mu^P} dZ_t^A. \qquad (48)
$$

Now, the dynamics of $\underline{u}_t$ becomes

$$
\begin{aligned}
d\underline{u}_t = & -\frac{1}{N_t}dN_t + \frac{1}{2N_t^2}\left\langle dN_t, dN_t\right\rangle - \frac{1}{A_t}dA_t + \frac{1}{2A_t^2}\left\langle dA_t, dA_t\right\rangle \\
& - \left(\frac{1-\beta}{\beta}\right)\left(\frac{1}{r-\mu_t^P}\right)d\mu_t^P - \left(\frac{1-\beta}{\beta}\right)\left(\frac{1}{2\left(r-\mu_t^P\right)^2}\right)\left\langle d\mu_t^P, d\mu_t^P\right\rangle \qquad (49)
\end{aligned}
$$

Let $\sigma_t^{\underline{u}}$ denote the diffusion of $\underline{u}_t$. By collecting the coefficients on $dZ_t^A$ in Equation (49), we have

$$
\sigma_t^{\underline{u}} = -\frac{\sigma_t^N}{N_t} - \sigma^A - \left(\frac{1-\beta}{\beta}\right)\left(\frac{\sigma_t^{\mu^P}}{r-\mu_t^P}\right), \qquad (50)
$$

which, in comparison with Equation (46), contains an extra term that reflects the volatility of expected token price change. Note that, similar to Equation (45), we have

$$
dN_t = -g\left(\underline{u}_t\right)d\underline{u}_t - \frac{1}{2}g'\left(\underline{u}_t\right)\left\langle d\underline{u}_t, d\underline{u}_t\right\rangle, \qquad (51)
$$

so the diffusion of $N_t$ is $-g\left(\underline{u}_t\right)\sigma_t^{\underline{u}}$. Matching it with the conjectured diffusion $\sigma_t^N$, we can solve $\sigma_t^N$.

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